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ON THE VOLUME FROM PLANAR SECTIONS THROUGH A CURVE

XIMO GUAL-ARNAU

Department of Mathematics, Campus Riu Sec, s/n, University Jaume I, 12071-Castelló, Spain
e-mail: gual@mat.uji.es
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ABSTRACT

We derive a formula to obtain the volume of a compact domain from planar sections through a curve. From this formula we propose a stereological estimator for the volume which generalizes some known unbiased estimators which use a systematic sampling scheme. Moreover we formulate a Cavalieri's principle for compact domains is spaces of constant curvature λ.

Keywords: Cavalieri, curve, planar sections, space form, unbiased estimator, volume.

INTRODUCTION

A problem of biomedical interest is how to estimate the volume of an object (bladder, prostate, ...) from planar sections. The best known method to obtain an unbiased estimation of this volume is provided by the Cavalieri's principle, which is based on parallel equidistant plane sections; that is, orthogonal plane sections through a line. The Cavalieri's method has been used for scanning techniques such as computed tomography (Pache *et al.*, 1993) or magnetic resonance imaging (Roberts *et al.*, 1994). However, it is difficult to obtain parallel sections of a given object (organ) by ultrasound scan when such obstacles as bones or air are present between the scanner and the organ. In this case, in Watanabe (1982) the volume of an object in $\mathbb{R}^3$ is calculated from planar sections through a curve of centroids. From this result, a numerical method has been developed to approximate the volume of an object, whose accuracy has been proved in Treece *et al.* (1999). However, an unbiased estimator of the volume is not available from this method and only from scanning planes which emanate from a common axis, an unbiased estimator for the volume has been derived, based on the ancient theorem of Pappus of Alexandria (Cruz-Orive and Roberts, 1993).

Here we calculate the volume of a compact domain in $\mathbb{R}^n$ from plane sections through a curve $c(t)$. When $c(t)$ is a line and the plane sections are orthogonal to the curve we obtain the Cavalieri method; when $c(t)$ is a curve of centroids of the plane sections we obtain the formula in Watanabe (1982) and when $c(t)$ is a circumference and the plane sections are normal to $c(t)$, these plane sections are rotating planes that contain a vertical axis and we have the result in Cruz-Orive and Roberts (1993). Moreover, from this formula we obtain an unbiased estimator for the volume when systematic sampling through the curve is considered. Stereological estimators of volume, using a systematic sampling scheme, are usually much more precise than those based on independent sampling. Therefore, systematic sampling is widely used in stereology as sampling scheme. Here we will consider systematic sampling through a curve to obtain unbiased estimators for the volume.

Since our principal aim is to present the mathematical foundations to derive formulas to calculate the volume from plane sections through a curve, in section "Volume of domains in $\mathbb{R}^n$" we derive the general formula for domains in $\mathbb{R}^n$ and we obtain some important consequences of this formula. However, in section "Stereological applications" we will concentrate on the applications in volume and area estimation in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, and we add two examples of bodies with sectional planes orthogonal to different curves. In section "Discussion" we generalize some results for compact domains in a space form M^n_λ of dimension n and sectional curvature λ, and we formulate a Cavalieri's principle in M^n_λ.

VOLUME OF DOMAINS IN $\mathbb{R}^n$

Let D be a compact domain in the n-dimensional euclidean space $\mathbb{R}^n$. Let $c : I = [0,L] \longrightarrow \mathbb{R}^n$ be a C^∞ curve parametrized by its arc-length t ($c(t)$ may be inside D or not). For every $t \in I$, let P_t denote a $(n-1)$-dimensional plane (hyperplane) in $\mathbb{R}^n$ through $c(t)$ (not necessarily orthogonal to $c(t)$). Let $\{E_1(t), E_2(t), \ldots, E_n(t)\}$ be a smooth orthonormal frame along $c(t)$ such that, for each $t \in I$, $\{E_2(t), E_3(t), \ldots, E_n(t)\}$ is a basis of $T_{c(t)}P_t \equiv P_t$.

From now on we will suppose that there exist subsets $D_t \subset P_t \cap D$ such that

$$D = \bigcup_{t \in I} D_t \quad \text{and} \quad D_{t_1} \cap D_{t_2} = \emptyset \quad \text{for} \quad t_1 \neq t_2 \ . \quad (1)$$

Under the above hypotheses, each point $x_t \in D_t$ will be given as

$$x_t = c(t) + \sum_{i=2}^{n} \mu_i(t) E_i(t) \; . \qquad (2)$$

Let $\mu(t) = \sum_{i=2}^{n} \mu_i(t) E_i(t) = \sum_{i=2}^{n} \langle \mu(t), E_i(t) \rangle E_i(t)$ and $N(t) = \mu(t)/|\mu(t)|$; then,

$$x_t = c(t) + r_t N(t) \; , \qquad (3)$$

where $r_t = dist(c(t), x_t)$.

Theorem 1. The volume of D is given by

$$Vol_n(D) = \int_0^L Vol_{n-1}(D_t) \langle c'(t), E_1(t) \rangle dt - \int_0^L \int_{D_t} r_t \langle N(t), \frac{d}{dt} E_1(t) \rangle \sigma_t \, dt \; , \qquad (4)$$

where σ_t is the $(n-1)$-dimensional volume element of P_t.

Proof. Let ω be the volume element in $\mathbb{R}^n$ and we consider on D the coordinates given by $(t, \mu_2(t), \ldots, \mu_n(t))$; then,

$$Vol_n(D) = \int_D \omega = \int_0^L \int_{D_t} \omega(\partial t, E_2(t), \ldots, E_n(t)) \sigma_t dt \; . \qquad (5)$$

From the properties of the cross vector product we get

$$Vol_n(D) = \int_0^L \int_{D_t} \langle \partial t, E_2(t) \wedge \cdots \wedge E_n(t) \rangle \sigma_t \, dt \; . \qquad (6)$$

But

$$\partial t = \left. \frac{\partial}{\partial t} \right|_{x_t} = \frac{d}{dt}(c(t) + r_t N(t)) = c'(t) + r_t N'(t) \; , \qquad (7)$$

so

$$\begin{aligned} \langle \partial t, E_2(t) \wedge \cdots \wedge E_n(t) \rangle &= \langle \partial t, E_1(t) \rangle \\ &= \langle c'(t), E_1(t) \rangle + r_t \langle N'(t), E_1(t) \rangle \\ &= \langle c'(t), E_1(t) \rangle - r_t \langle N(t), \frac{d}{dt} E_1(t) \rangle \; . \qquad (8) \end{aligned}$$

Now, substituting Eq. 8 in Eq. 6 we obtain the result.

In order to obtain some important consequences of the above theorem we will recall the concept of moment and center of masses of a domain.

MOMENTS AND CENTER OF MASSES (OR CENTROID) OF D_t

Let Γ be a $(n-2)$-dimensional plane (that is, an hyperplane in P_t) through $c(t)$, with unit normal vector field ξ. Γ separates $P_t - \Gamma$ into two components. Let A be the component ξ points to. Let ε be the real function defined on P_t by

$$\varepsilon(x_t) = \begin{cases} 1 & \text{if } x_t \in A \\ -1 & \text{if } x_t \notin A \end{cases} \qquad (9)$$

The moment of D_t respect to Γ ($M_\Gamma(D_t)$) is given by the integral

$$M_\Gamma(D_t) = \int_{D_t} \varepsilon(x_t) l(x_t) \sigma_t \; , \qquad (10)$$

where l is the distance from x_t to Γ.

From elementary trigonometric properties we have that

$$M_\Gamma(D_t) = \int_{D_t} r_t \langle \xi, N(t) \rangle \sigma_t \; . \qquad (11)$$

Hence, $c(t)$ is the centroid of D_t if for every unit vector $\xi \in T_{c(t)} P_t \equiv P_t$, one has that $M_\Gamma(D_t) = 0$.

Now we come back to Theorem 1. Suppose that

$$\frac{d}{dt} E_1(t) = \sum_{i=1}^{n} a_i(t) E_i(t) \qquad (12)$$

and let Γ_i denote the hyperplane orthogonal to $E_i(t)$ in P_t, $(i = 2, 3, \ldots, n)$.

Corollary 1. The volume of D is given by

$$Vol_n(D) = \int_0^L Vol_{n-1}(D_t) \langle c'(t), E_1(t) \rangle dt - \sum_{i=1}^{n} \int_0^L a_i(t) M_{\Gamma_i}(D_t) \, dt \; . \qquad (13)$$

Proof. Immediate from (12) and Theorem 1.

Now we will consider some important consequences of the above corollary.

$c(t)$ is a curve of centroids. (See Figs. 1, 2)

Suppose that, for each $t \in [0, L]$, $c(t)$ is the center of masses of D_t; then, from Eq. 13,

$$Vol_n(D) = \int_0^L Vol_{n-1}(D_t) \langle c'(t), E_1(t) \rangle \, dt \; . \qquad (14)$$

The above formula for $n = 3$ is a generalization of the main result in (Watanabe, 1982); however the method used here to obtain it is completely different.

P_t is orthogonal to c. (See Figs. 2, 5)

Suppose that for each $t \in [0,L]$, P_t is an orthogonal hyperplane to $c(t)$. Now, we shall consider that the curve $c(t)$ has a Frenet frame $\{f_1(t) = c'(t), f_2(t), \dots, f_n(t)\}$, which is positively oriented and satisfies the Frénet equations:

$$\begin{aligned} f_1'(t) &= k_1(t) f_2(t) \\ f_2'(t) &= -k_1(t) f_1(t) + k_2(t) f_3(t) \\ &\vdots \\ f_{n-1}'(t) &= -k_{n-2}(t) f_{n-2}(t) + k_{n-1}(t) f_n(t) \\ f_n'(t) &= -k_{n-1}(t) f_{n-1}(t) \ , \end{aligned} \tag{15}$$

where $k_i(t)$ is called the ith curvature of $c(t)$.

Then, from Corollary 1, substituting $E_1(t)$ by $f_1(t)$ we obtain

$$Vol_n(D) = \int_0^L Vol_{n-1}(D_t)dt - \int_0^L k_1(t) M_{\Gamma_2}(D_t)dt \ , \tag{16}$$

where $M_{\Gamma_2}(D_t)$ is the moment of D_t respect to the hyperplane in P_t orthogonal to $f_2(t)$.

Corollary 2. When $c(t)$ is a straight line in $\mathbb{R}^n$, ($k_i(t) = 0$), or when $M_{\Gamma_2}(D_t) = 0$, we obtain the Cavalieri's formula

$$Vol_n(D) = \int_0^L Vol_{n-1}(D_t)dt \ . \tag{17}$$

From Eq. 9 and Eq. 10, if $D_t^+ = D_t \cap A$ and $D_t^- = \{x_t \in D_t \ / \ x_t \notin A\}$, we have that $M_{\Gamma_2}(D_t) = 0$ means

$$\int_{D_t^+} l(x_t)\sigma_t = \int_{D_t^-} l(x_t)\sigma_t \ , \tag{18}$$

that is, the distance from the centroid of D_t^- to Γ_2 and the distance from the centroid of D_t^+ to Γ_2 coincide.

STEREOLOGICAL APPLICATIONS

In this section we will concentrate on the stereological applications of volume estimation in $\mathbb{R}^3$ and area estimation in $\mathbb{R}^2$.

Let D be a compact domain in $\mathbb{R}^3$ and $c(t)$ a curve which satisfy the conditions imposed in Eq. 1 and such that P_t is the orthogonal plane to $c(t)$. We define

$$f(t) = Area(D_t) - k(t) M_2(D_t) \ , \tag{19}$$

where $k(t)$ is the curvature of $c(t)$ and $M_2(D_t)$ is the moment of D_t with respect to the line in P_t given by the binormal vector of the Frenet frame in $c(t)$.

Then, $Vol(D)$ can be expressed as an integral $Vol(D) = \int_0^L f(t)dt$ and may be estimated from systematic sampling on $[0,L]$; that is, let $T = L/m$ and t_0 a point placed uniformly at random in $[0,T]$, then

$$\widehat{V}(t_0) = T \sum_{i=0}^{m-1} f(t_0 + iT) \tag{20}$$

is an unbiased estimator of $Vol(D)$.

Since c is parametrized by its arc -length it is locally injective. If we suppose that c is globally injective (c has no self-intersections), then c is an isometry and therefore the distance between $c(t_0 + iT)$ and $c(t_0 + (i-1)T)$ is T; that is, the sets D_{iT} are orthogonal to the curve and placed at equidistant intervals with spacing T along the curve. Moreover, the point $\alpha(t_0)$ is placed uniformly at random in $[\alpha(0), \alpha(T)]$.

It is important to note that the torsion of c does not appear in Eq. 20. Properties of this kind of estimators (variance, etc.) have been widely considered in the literature (see, *e.g.*, Gual-Arnau and Cruz-Orive (1998)).

Note that when $c(t)$ is a curve of centroids or when $c(t)$ is a line ($k(t) = 0$) the measure function $f(t)$ is given by $f(t) = Area(D_t)$.

For a compact domain D in $\mathbb{R}^2$, $f(t)$ has the form

$$f(t) = Length(D_t) - k(t) M_{c(t)}(D_t) \ , \tag{21}$$

with

$$M_{c(t)}(D_t) = \int_{D_t} \varepsilon(x_t) r_t \eta_t \ , \tag{22}$$

where η_t is the line element of P_t and $\varepsilon(x_t)$ is given by Eq. 9.

Now we will prove that the method developed in Eq. 3.3 of Cruz-Orive and Roberts (1993), to obtain the volume of a domain in $\mathbb{R}^3$ from scanning planes which emanate from a common axis Oz, is a particular case of our method when the curve $c(t)$ is a circumference of unit radius placed in a plane orthogonal to Oz.

Using the notation in Cruz-Orive and Roberts (1993) and supposing that D_t lies entirely in the half space A, the moment $M_{\Gamma_2}(D_t)$ can be written as:

$$M_2(D_t) = -l_1^+(t) Area(D_t) \ , \tag{23}$$

where $l_1^+(t)$ is the distance from Γ_2 to the centroid of D_t. Then, from Eq. 21 and Eq. 23, we have

$$Vol(D) = \int_0^{2\pi} Area(D_t)dt + \int_0^{2\pi} l_1^+(t)Area(D_t)dt$$
$$= \int_0^{2\pi} Area(D_t)(1 + l_1^+(t))dt = \int_0^{2\pi} l_2^+(t)Area(D_t)dt, \tag{24}$$

where $l_2^+(t)$ is the distance from Oz to the centroid of D_t, which is the Eq. 3.3 in Cruz-Orive and Roberts (1993).

To finish this section we will present two examples with schematic figures of different domains, curves and planes sections.

In the first example the curve $c(t)$ is a curve of centroids which is inside D.

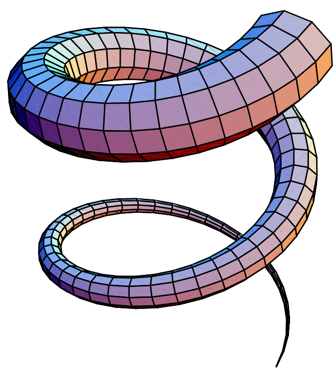

Fig. 1. *Domain D.*

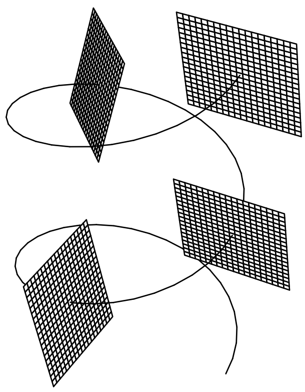

Fig. 2. *Curve of centroids and portions of planes P_t orthogonal to the curve.*

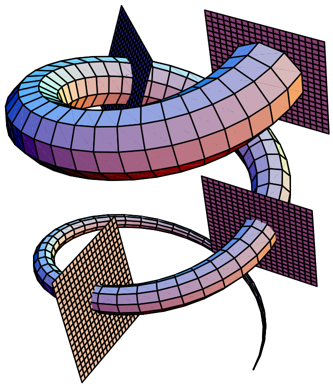

Fig. 3. *Domain with some plane sections which give D_t.*

In the second example the curve $c(t)$ is outside D and the scheme is similar to that of Cavalieri.

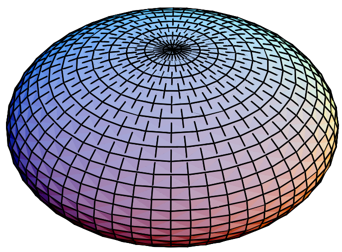

Fig. 4. *Domain D.*

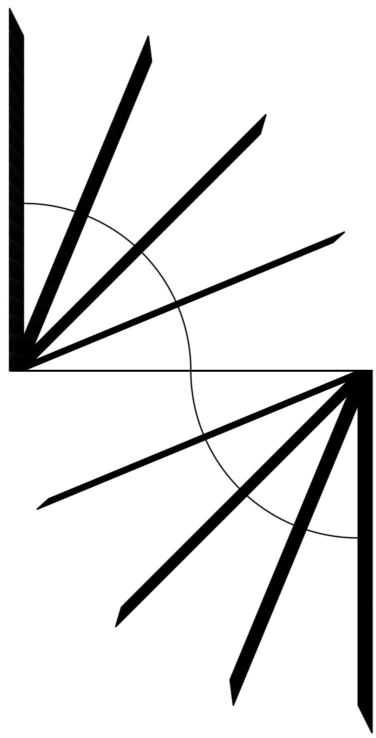

Fig. 5. *Curve $c(t)$ and planes P_t vertical to the $x-y$ plane.*

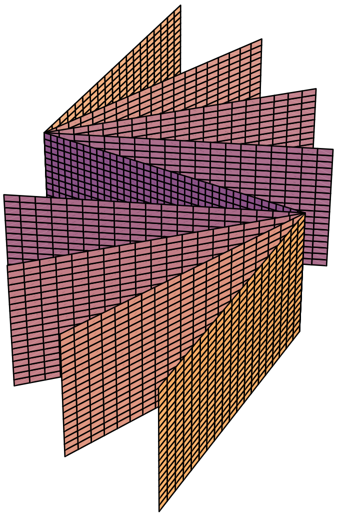

Fig. 6. *Planes P_t from another viewpoint.*

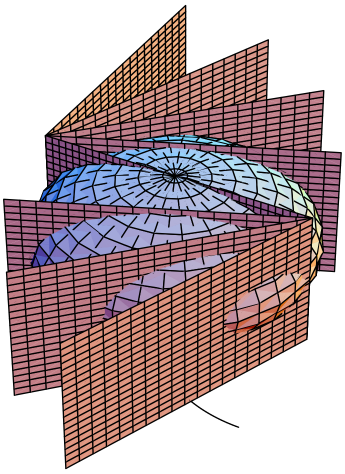

Fig. 7. *Domain with some plane sections and the curve $c(t)$.*

DISCUSSION

Stereological estimators of volume using a systematic scheme, based on equidistant cuts through a line, are usually much more precise than those based on independent sampling; because any systematic sample represents the whole material better than most samples obtained by independent sampling. Our estimator Eq. 20 preserves the systematic sampling scheme and offers the possibility to choose an alternative curve to a line whose orthogonal sample planes represent better the whole material.

On the other hand, our approach may be adapted to several clinical areas; for example, in the vessel study from IVUS (Intravascular Ultrasound) images, an artery can be modelled as a tube of non-constant section (Fig. 3 is a particular case of these tubes), where the catheter trajectory is the curve c.

When the measurement function $f(t)$ given in Eq. 19 is not known, its estimation will depend on the information provided in each practical case.

VOLUME OF DOMAINS IN A SPACE FORM

Now we will extend Eq. 16 for compact domains D in a simply connected space form M^n_λ of dimension n and sectional curvature λ, with $\lambda \neq 0$ (for $\lambda = 0$ we have the results in Section 2). Let $c : I = [0,L] \longrightarrow M^n_\lambda$ be a C^∞ curve parametrized by its arc-length t and, for every $t \in [0,L]$, let P_t denote a complete totally geodesic hypersurface of M^n_λ through $c(t)$ and orthogonal to the curve c. We suppose that there exists a Frenet frame along c with the same properties as in Eq. 15 but $f_i'(t)$ means, now, the covariant derivative of $f_i(t)$ along $c(t)$, $\left(\frac{\nabla}{dt} f_i(t)\right)$.

Under the same assumptions as in Eq. 1, each point $x_t \in D$ will be given by

$$x_t = \exp_{c(t)} r_t N(t) = \exp_{c(t)} \mu(t) \; , \tag{25}$$

where

$$\mu(t) = \sum_{i=2}^{n} \mu_i(t) F_i(t) \; . \tag{26}$$

Supposing now that $D \subset U$, where U is the image by exp of an open set of the normal bundle of c on which exp is a diffeomorphism, we may consider on U the coordinates:

$$\phi(\exp_{c(t)} r_t N(t)) = (t, \mu_2(t), \ldots, \mu_n(t)) \; , \tag{27}$$

then

$$Vol_n(D) = \int_0^L \int_{D_t} \langle \partial t, \tau_r f_2(t) \wedge \cdots \wedge \tau_r f_n(t) \rangle \sigma_t \, dt \; , \tag{28}$$

where τ_r is the parallel transport from $c(t)$ to $\gamma_{N(t)}(r_t)$ ($\gamma_{N(t)}$ is the geodesic with $\gamma_{N(t)}(0) = c(t)$ and $\gamma'_{N(t)}(0) = N(t)$).

Now,

$$\partial t = \frac{d}{dt}(\exp_{c(t)} r_t N(t)) = Y_1(r_t) \; , \tag{29}$$

where Y_1 is the Jacobi field along $\gamma_{N(t)}$ satisfying:

$$Y_1(0) = f_1(t), \quad Y_1'(0) = \frac{\nabla}{dt} N(t) \; , \tag{30}$$

which is given by Gray and Miquel (2000)

$$Y_1(r_t) = c_\lambda(r_t)\tau_r f_1(t) + s_\lambda(r_t)\tau_r \frac{\nabla}{dt}N(t) \ , \qquad (31)$$

where, for every $\lambda \in \mathbb{R}$, $s_\lambda : \mathbb{R} \to \mathbb{R}$ will denote the solution of the equation $s'' + \lambda s = 0$ with the initial conditions $s(0) = 0$ and $s'(0) = 1$; and $c_\lambda = s'_\lambda$; i. e.

$$s_\lambda(s) = \begin{cases} \sin(s\sqrt{\lambda})/\sqrt{\lambda}, & \lambda > 0 \\ s, & \lambda = 0 \\ \sinh(s\sqrt{\lambda})/\sqrt{\lambda}, & \lambda < 0 \ . \end{cases} \qquad (32)$$

(Note that for $\lambda = 0$, $Y_1(r_t)$ is given by Eq. 7).

Finally, from Eq. 28 and Eq. 31 we have that

$$\begin{aligned} Vol_n(D) = & \int_0^L \left(\int_{D_t} c_\lambda(r)\sigma_t \right) dt \\ & + \int_0^L \int_{D_t} s_\lambda(r_t) \left\langle \frac{\nabla}{dt}N(t), f_1(t) \right\rangle \sigma_t \, dt \\ = & \int_0^L \left(\int_{D_t} c_\lambda(r)\sigma_t \right) dt - \int_0^L M_{\Gamma_2}(D_t)k_1(t)dt \ , \quad (33) \end{aligned}$$

where

$$M_{\Gamma_2}(D_t) = \int_{D_t} s_\lambda(r_t)\langle f_2(t), N(t)\rangle \sigma_t \ , \qquad (34)$$

and Eq. 33 generalizes Eq. 16 for space forms M^n_λ.

In the particular case where the compact domain D is obtained by a motion of a domain D_0 through the curve $c(t)$; Eq. 33 has been obtained in Gray and Miquel (2000).

ABOUT THE CAVALIERI PRINCIPLE IN M^n_λ

The Cavalieri principle, already familiar to the ancient Greeks, states that the volumes of two solids in $\mathbb{R}^n$ are equal if the areas of the corresponding sections drawn perpendicular to a straight line are equal. This principle is not valid when totally geodesic hypersurfaces perpendicular to a given geodesic are considered; however, from Eq. 33 it is possible to formulate this principle for compact domains in M^n_λ as follows:

Volumes of two compact domains in M^n_λ are equal if the integrals

$$\int_{D_t} c_\lambda(r)\sigma_t \qquad (35)$$

are equal through a geodesic $c(t)$ in M^n_λ where r denotes the geodesic distance from x to $c(t)$.

(The integral Eq. 35 is equal to $Vol_{n-1}(D_t)$ for $\lambda = 0$ and it is called the 'modified volume' in Choe and Gulliver (1992).)

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