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A NEW FRACTIONAL APPROACH FOR THE HIGHER-ORDER q-TAYLOR METHOD

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ABSTRACT

The main goal of this work is to propose a new fractional approach of the higher-order q-Taylor method with Initial Value Problems (IVPs) for fractional q-difference equations which is called the Fractional Higher-Order q-Taylor Method (FHOqTM). By applying the generalised q-Taylor theorem, this would be achieved. As a consequence, we calculate the FHOqTM's local truncation error. Finally, we present numerical applications to validate our results by comparing the exact solution and the approximate solution obtained by (FHOqTM).

Keywords: Caputo q-derivative, Fractional q-difference equations, Generalized q-Taylor theorem, Higher-order q-Taylor method.

INTRODUCTION

Fractional q-calculus is an interesting topic and an important branch in mathematical analysis, which was first established and developed in the 20th century by Jackson (1910; 1908), Al-Salam (1966-1967) and Agarwal (1969). It has been of interest to many academics because of its application to mathematical modeling in several fields, including biomathematics, engineering, physics, and technical sciences. Furthermore, fractional q-difference equations have also played an essential role in modeling a range of phenomena in many areas; for more specifics see Abdeljawad and Baleanu (2012); Ahmad *et al.* (2012); Annaby and Mansour (2012); Kac and Cheung (2002); Rajkovic *et al.* (2007a;b).

In recent years, the initial and boundary value problems of fractional q-difference equations involving Caputo's fractional q-derivative received a lot of interest from scholars, they studied and discussed solutions to these problems. There are two types of solutions, the first type is analytical solutions, which involve using traditional and analytical techniques to solve problems, but scientists faced difficulties and barriers in doing so; for details, see the references Abbas *et al.* (2021; 2019); Abdeljawad and Baleanu (2012); Allouch and Hamani (2023; 2024); Allouch *et al.* (2022); Boutiara *et al.* (2021). This encouraged consideration of the second type, numerical solutions, which involve numerical techniques and applications. However, they did not develop and progress much in them because researching these type of fractional q-difference equations is a modern and contemporary approach; see Hamadneh *et al.* (2023); Samei (2019); Samei and Yang (2020) and references therein.

There are several numerical methods and techniques, including the Taylor method that caught our interest, which is an algorithm used to estimate the solution of equations. After that, scientists became interested and developed Taylor's method and used it to find approximate solutions to fractional differential equations, for more details; see Barrio (2005); Barrio *et al.* (2005); Batiha *et al.* (2023a;b); Zaid and Momani (2008); Usero (2008).

In fractional q-calculus, researchers generalized and investigated Taylor's technique, it is called generalized q-Taylor's Method which includes Caputo's fractional q-derivative, for more information see the references Garg *et al.* (2013); Hassan (2016); Sana *et al.* (2021). It has attracted great attention from scholars, as they have used in order to identify estimate solutions to fractional q-difference equations.

This research is to present a novel fractional approach for fractional q-difference equations, it's calles the Fractional Higher-Order q-Taylor Method (FHOqTM), which is a higher order q-Taylor method and an algorithm used to estimate numerical solutions to initial value problem of fractional q-difference equations involving the Caputo's q-derivative of the form:

$$\begin{cases} {}^{C}D_q^{\gamma}x(t) = f(t, x(t)),\ 0 < \gamma \le 1,\ t \in [0,T], \\ x(0) = x_0. \end{cases} \quad (1)$$

where $q \in (0,1)$, $T > 0$ and $x_0 \in R$, ${}^{C}D_q^{\gamma}$ represents the Caputo's fractional q-derivative of order $\gamma \in (0,1]$ and $f : [0,T] \times R \to R$ is continuous function.

The remainder of the document is structured as follows: In Section 2, we provide several foundational definitions and properties of fractional q-calculus and give q-Taylor's generalized theorem. In Section 3, we establish the Fractional Higher-Order q-Taylor Method (FHOqTM) to solve the initial value problem of fractional q-difference equations wich is based on generalized q-Taylor's theorem, we will discuss and estimate the local truncation error produced by FHOqTM by presenting some theoretical results. Finally, we present some numerical examples in Section 4 to demonstrate the applicability of the suggested approache.

BASIC FACTS

This section reviews the fundamental definitions and some properties for fractional q-calculus, for details see Garg *et al.* (2013); Kac and Cheung (2002); Rajkovic *et al.* (2007a;b).
For $q \in (0,1)$, we define:

$$[a]_q = \frac{1-q^a}{1-q}; \quad a \in R.$$

Let $a, b \in R$ and $n \in N$, the q-analogue of the power $(a-b)^{(n)}$ is expressed by:

$$(a-b)^{(n)} = \prod_{i=0}^{n-1}(a - bq^i), \quad (a-b)^{(0)} = 1.$$

In generally, for $a, b, \gamma \in R$, we have:

$$(a-b)^{(\gamma)} = a^{\alpha} \prod_{i=0}^{\infty} \left(\frac{a - bq^i}{a - bq^{i+\gamma}} \right),$$

Note that, if $b = 0$, then $a^{(\gamma)} = a^{\gamma}$.

Definition 1 *Kac and Cheung (2002) For all $\gamma \in R_+^*$. The q-gamma function is determined as follows:*

$$\Gamma_q(\gamma) = \frac{(1-q)^{(\gamma-1)}}{(1-q)^{\gamma-1}}.$$

Note that the q-gamma function satisfy

$$\Gamma_q(\gamma+1) = [\gamma]_q \Gamma_q(\gamma).$$

Definition 2 *Kac and Cheung (2002) Let f be a function defined on $[0,T]$ and $n \in N$. The q-derivative of order n is defined by :*

$$(D_q f)(t) = (D_q^1 f)(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0,$$

$$(D_q f)(0) = \lim_{t \to 0}(D_q f)(t),$$

and

$$(D_q^n f)(t) = (D_q^1 D_q^{n-1} f)(t), \quad n = 1, 2, \ldots.$$

Definition 3 *Rajkovic* et al. *(2007a) Let $f : [0,T] \to R$ and for all $\gamma \ge 0$. The Riemann-Liouville's fractional q-integral of order γ is given by:*

$$(I_q^{\gamma} f)(t) = \frac{1}{\Gamma_q(\gamma)} \int_0^t (t - qs)^{(\gamma-1)} f(s) d_q s.$$

The fractional q-integral of Riemann-Liouville has several properties, including Rajkovic *et al.* (2007a):
1) $(\mathrm{I}_q^0 f)(t) = f(t)$.
2) $(I_q^{\gamma} t^{(\lambda)})(t) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\gamma+\lambda+1)} t^{(\gamma+\lambda)},\ \gamma, \lambda \in R_+$.
3) $(I_q^{\gamma} I_q^{\lambda} f)(t) = (I_q^{\gamma+\lambda} f)(t) = (I_q^{\beta} I_q^{\alpha} f)(t),\ \gamma, \lambda \in R_+$.

Definition 4 *Rajkovic* et al. *(2007b) Let $f : [0,T] \to R$ and for $\gamma \ge 0$. The Caputo's fractional q-derivative of order γ is given by:*

$$({}^{C}D_q^{\gamma} f)(t) = (I_q^{[\gamma]-\gamma} D_q^{[\gamma]} f)(t),$$

where $[\gamma]$ represents the integer part of γ.

The Caputo's fractional q-derivative has the following properties Rajkovic *et al.* (2007b):
1) $({}^{C}D_q^{0} f)(t) = f(t)$.
2) ${}^{C}D_q^{\gamma} c = 0,\ c \in R$.
3) $({}^{C}D_q^{\gamma} t^{(\lambda)})(t) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\gamma+1)} t^{(\lambda-\gamma)},\ \gamma, \lambda \in R_+$.

Now, we introduce the generalized q-Taylor's formulat theorems.

Theorem 1 *Garg* et al. *(2013)*(**q-Taylor's Generalized Theorem**) *Assume that ${}^{C}D_q^{j\gamma} f(t) \in C([0,T])$ for $j = 0, 1, 2, \ldots, n+1$, where $q \in (0,1)$ and $\gamma \in (0,1]$. Then the function f can be extended as follows about $t = t_0$:*

$$f(t) = \sum_{i=0}^{n} \frac{(t-t_0)^{(i\gamma)}}{\Gamma_q(i\gamma+1)}\, {}^{C}D_q^{i\gamma} f(t_0) + \frac{(t - qt_0)^{((n+1)\gamma)}}{\Gamma_q((n+1)\gamma+1)}\, {}^{C}D_q^{(n+1)\gamma} f(\xi), \quad (2)$$

with $0 < \xi \le t$, $\forall t \in [0,T]$.

RESULTS AND DISCUSSION

Our goal in this part is to establish a fractional higher-order q-Taylor Method (FHOqTM) version. The generalised q-Taylor method's principles will be used to accomplish this. In addition, we will discuss the precise local truncation error of our developed methodology.

To establish our main results for the problem (1), we present the following lemma:

Lemma 1 *The fractional initial value problem (1) has an approximate solution given as follows:*

$$\begin{cases} v_0 = x_0, \\ v_{i+1} v_i + h^{\gamma} T(t_i, v_i), \end{cases} \tag{3}$$

with

$$\begin{aligned} T(t_i,v_i) &= \frac{1}{\Gamma_q(\gamma+1)} f(t_i,v_i) + \frac{h^{\gamma}}{\Gamma_q(2\gamma+1)}\, {}^{C}D_q^{\gamma} f(t_i,v_i) \\ &\quad + \cdots + \frac{h^{(n-1)\gamma}}{\Gamma_q(n\gamma+1)}\, {}^{C}D_q^{(n-1)\gamma} f(t_i,v_i), \end{aligned} \tag{4}$$

for $i = 0,1,\ldots,n-1$, such that v_i represents the approximate solution of the exact solution x at t_i and h represents the step size.

***Proof* 1** *To show the result, we divided interval $[0,T]$ as follows:*

$$0 = t_0 < t_1 = t_0 + h < t_2 = t_0 + 2h \cdots < t_n = t_0 + nh = b,$$

in which the mesh points $t_i = t_0 + ih$, $i = 1,2,\ldots,n$, with $h = \frac{b}{n}$ is the step size. Suppose that

$${}^{C}D_q^{(n+1)\gamma} y(t) \in C^{n+1}([0,T]).$$

Now, by applying q-Taylor's generalised theorem mentioned in Theorem 1. The solution $x(t)$ can be expanded about $t = t_i$ as:

$$\begin{aligned} x(t) &= x(t_i) + \frac{{}^{C}D_q^{\gamma} x(t_i)}{\Gamma_q(\gamma+1)} (t-t_i)^{(\gamma)} + \frac{{}^{C}D_q^{2\gamma} x(t_i)}{\Gamma_q(2\gamma+1)} \\ &\quad \times (t-t_i)^{(2\gamma)} + \cdots + \frac{{}^{C}D_q^{n\gamma} x(t_i)}{\Gamma_q(n\gamma+1)} (t-t_i)^{(n\gamma)} \\ &\quad + \frac{{}^{C}D_q^{(n+1)\gamma} x(\xi)}{\Gamma_q((n+1)\gamma+1)} (t-t_i)^{((n+1)\gamma)}, \end{aligned}$$

where $t_i < \xi < t_{i+1}$.
By changing t to t_{i+1} in the previous equality, we find:

$$\begin{aligned} x(t_{i+1}) &= x(t_i) + \frac{h^{\gamma}}{\Gamma_q(\gamma+1)}\, {}^{C}D_q^{\gamma} x(t_i) + \frac{h^{2\gamma}}{\Gamma_q(2\gamma+1)} \\ &\quad \times {}^{C}D_q^{2\gamma} x(t_i) + \cdots + \frac{h^{n\gamma}}{\Gamma_q(n\gamma+1)}\, {}^{C}D_q^{n\gamma} x(t_i) \\ &\quad + \frac{h^{(n+1)\gamma}}{\Gamma_q((n+1)\gamma+1)}\, {}^{C}D_q^{(n+1)\gamma} x(\xi). \end{aligned} \tag{5}$$

Because of,

$$\begin{aligned} {}^{C}D_q^{\gamma} x(t) &= f(t,x(t)), \\ {}^{C}D_q^{2\gamma} x(t) &= {}^{C}D_q^{\gamma} f(t,x(t)), \\ &\vdots \\ {}^{C}D_q^{n\gamma} x(t) &= {}^{C}D_q^{(n-1)\gamma} f(t,x(t)). \end{aligned}$$

This will transform formula (5) into the following:

$$\begin{aligned} x(t_{i+1}) &= x(t_i) + \Bigg[\frac{h^{\gamma}}{\Gamma_q(\gamma+1)} f(t_i,x(t_i)) \\ &\quad + \frac{h^{2\gamma}}{\Gamma_q(2\gamma+1)}\, {}^{C}D_q^{\gamma} f(t_i,x(t_i)) + \cdots \\ &\quad + \frac{h^{n\gamma}}{\Gamma_q+(n\gamma+1)}\, {}^{C}D_q^{(n-1)\gamma} f(t_i,x(t_i)) \Bigg] \\ &\quad + \frac{h^{(n+1)\gamma}}{\Gamma_q((n+1)\gamma+1)}\, {}^{C}D_q^{n\gamma} f(\xi,x(\xi). \end{aligned} \tag{6}$$

In fact, formula (6) can be approximately represented as follows:

$$\begin{cases} v_0 = x_0, \\ v_{i+1} v_i + h^{\gamma} T(t_i, v_i), \end{cases}$$

with

$$\begin{aligned} T(t_i,v_i) &= \frac{1}{\Gamma_q(\gamma+1)} f(t_i,v_i) + \frac{h^{\gamma}}{\Gamma_q(2\gamma+1)}\, {}^{C}D_q^{\gamma} f(t_i,v_i) \\ &\quad + \cdots + \frac{h^{(n-1)\gamma}}{\Gamma_q(n\gamma+1)}\, {}^{C}D_q^{(n-1)\gamma} f(t_i,v_i), \end{aligned} \tag{7}$$

for $i = 0,1,\ldots,n-1$, which is what we wanted to prove.

Theorem 2 *Assume that the initial value problem (1) can be approximated using the FHOqTM with a step size of h, and suppose ${}^{C}D_q^{j\gamma} x(t) \in C([0,T])$, for $j = 0,1,2,\ldots,n+1$, where $q \in (0,1)$ and $\gamma \in (0,1]$. Then, The local truncation error is given by:*

$$O(h^{n\gamma}).$$

***Proof* 2** *According to formula (6), we can write:*

$$\begin{aligned} & x(t_{i+1}) - x(t_i) - \frac{h^{\gamma}}{\Gamma_q(\gamma+1)} f(t_i,x(t_i)) - \frac{h^{2\gamma}}{\Gamma_q(2\gamma+1)} \\ & \times {}^{C}D_q^{\gamma} f(t_i,x(t_i)) - \cdots - \frac{h^{n\gamma}}{\Gamma_q(n\gamma+1)}\, {}^{C}D_q^{(n-1)\gamma} f(t_i,x(t_i)) \\ & = \frac{h^{(n+1)\gamma}}{\Gamma_q((n+1)\gamma+1)}\, {}^{C}D_q^{n\gamma} f(\xi,x(\xi), \end{aligned}$$

for $\xi \in (t_i, t_{i+1})$. *Consequently, this gives:*

$$x(t_{i+1}) - x(t_i) - h^{\gamma} T(t_i, x(t_i)) \quad = \tfrac{h^{(n+1)\gamma}}{\Gamma_q((n+1)\gamma+1)} \, {}^{C}D_q^{n\gamma} f(\xi, x(\xi),$$

with

$$\begin{aligned} T(t_i, x(t_i)) \quad &= \quad \frac{1}{\Gamma_q(\gamma+1)} f(t_i, x(t_i)) + \frac{h^{\gamma}}{\Gamma_q(2\gamma+1)} \\ &\quad \times {}^{C}D_q^{\gamma} f(t_i, x(t_i)) + \cdots + \frac{h^{(n-1)\gamma}}{\Gamma_q(n\gamma+1)} \\ &\quad \times {}^{C}D_q^{(n-1)\gamma} f(t_i, x(t_i)), \end{aligned}$$

for $i = 0, 1, 2, \ldots, n-1$.
Consequently, the following is the local truncation error:

$$\mathscr{E}_{i+1}^{T}(h) = \frac{x(t_{i+1}) - x(t_i)}{h^{\gamma}} - T(t_i, x(t_i)).$$

This implies,

$$\mathscr{E}_{i+1}^{T}(h) = \frac{h^{n\gamma}}{\Gamma_q((n+1)\gamma+1)} \, {}^{C}D_q^{n\gamma} f(\xi, x(\xi)).$$

If ${}^{C}D_q^{j\gamma} x(t) \in C([0,T])$, *for* $j = 0, 1, 2, \ldots, n+1$, *then:*

$${}^{C}D_q^{(n+1)\gamma} x(t) = {}^{C}D_q^{n\gamma} f(t, x(t)),$$

which is limited on $[0,T]$. *Therefore,*

$$\mathscr{E}_{i+1}^{T}(h) = O(h^{n\gamma}).$$

ILLUSTRATIVE APPLICATIONS

This section aims to show the effectiveness of our proposed approach, which we call FHO*q*TM. Therefore, we present two numerical examples to illustrate our results, which we use FHO*q*TM in two different cases of fractional *q*-difference equations. Next, we highlight the results obtained and compare them with exact solutions by presenting them in graphs.

Example 1 *We consider the following initial value problem of fractional q-difference equations:*

$$\begin{cases} {}^{C}D_q^{\gamma} x(t) = -x(t) + t + 2, \; 0 < \gamma \le 1, \; t \in [0,1], \\ x(0) = \frac{1}{3}. \end{cases} \tag{8}$$

The exact solution to the problem (8) when $\gamma = 1$ *is given as:*

$$x(t) = t + 1 - \frac{2}{3} e^{-t}.$$

Here, we take $n = 100$, *so* $h = 0.01$. *Assume that:*

$$f(t, x(t)) = -x(t) + t + 2.$$

Additionally, let's suppose that we want to use on the FHOqTM of order 2γ. *To achieve this, we obtain:*

$${}^{C}D_q^{\gamma} f(t, x(t)) = x(t) - t - 2 + \frac{t^{(1-\gamma)}}{\Gamma_q(2-\gamma)}.$$

As a result, we can determine $T(t_i, v_i)$ *by using equation (7) as follows:*

$$\begin{aligned} T(t_i, v_i) \quad &= \quad \frac{1}{\Gamma_q(\gamma+1)} \Big(-v_i + t_i + 2 \Big) + \frac{h^{\gamma}}{\Gamma_q(2\gamma+1)} \\ &\quad \times \left(v_i - t_i - 2 + \frac{t_i^{(1-\gamma)}}{\Gamma_q(2-\gamma)} \right), \end{aligned} \tag{9}$$

where v_i *represent approximations for* $x(t_i)$ *for* $i = 0, 1, 2, \ldots, 99$.
Since $t_i = t_0 + ih = 0.01i$ *for* $i = 0, 1, 2, \ldots, 99$. *So, we can write the formula (9) as:*

$$\begin{aligned} T(t_i, v_i) \quad &= \quad \frac{1}{\Gamma_q(\gamma+1)} \Big(-v_i + 0.01i + 2 \Big) + \frac{(0.01)^{\gamma}}{\Gamma_q(2\gamma+1)} \\ &\quad \times \left(v_i - 0.01i - 2 + \frac{(0.01)i^{(1-\gamma)}}{\Gamma_q(2-\gamma)} \right). \end{aligned} \tag{10}$$

Consequently, by equation (10), we can express the 2γ *order of the FHOqTM in the following form:*

$$\begin{cases} v_0 = \frac{1}{3}, \\ v_{i+1} = v_i + (0.01)^{\gamma} \Big[\frac{1}{\Gamma_q(\gamma+1)} \Big(-v_i + 0.01i + 2 \Big) \\ \qquad + \frac{(0.01)^{\gamma}}{\Gamma_q(2\gamma+1)} \Big(v_i - 0.01i - 2 + \frac{(0.01)i^{(1-\gamma)}}{\Gamma_q(2-\gamma)} \Big) \Big], \end{cases} \tag{11}$$

for $i = 0, 1, 2, \ldots, 99$.
Considering the formula (11), we can simulate approximate solutions by applying the FHOqTM of order 2γ *with exact solution to the problem (8), it is shown in Fig. 1 for* $q = 0.99$, $\gamma = 1$ *and Fig. 2 represent the absolute errors of order* 2γ.

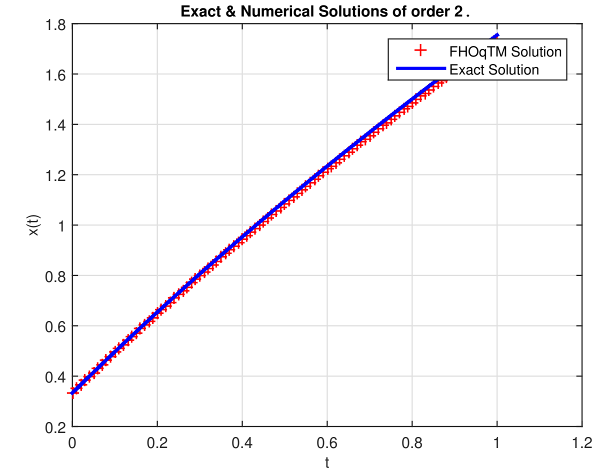

Fig. 1: *Exact & Numerical solutions of order* 2γ *for* $q = 0.99$ *and* $\gamma = 1$.

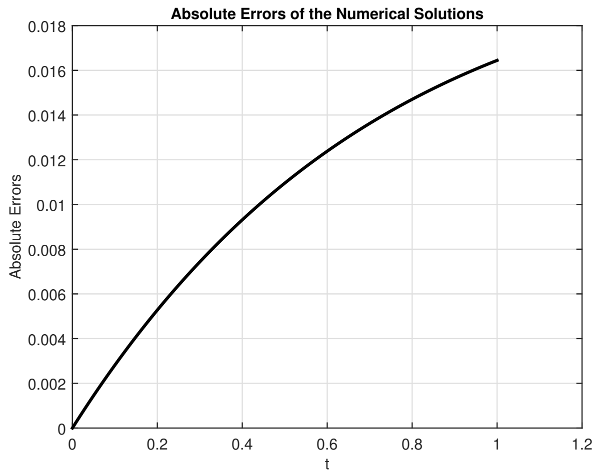

Fig. 2: *Absolute Errors of order of order* 2γ *for* $q = 0.99$ *and* $\gamma = 1$.

By taking different values of γ *for* $q = 0.99$*, various values of* q *for* $\gamma = 1$*, respectively. We plot in Fig. 3 and Fig. 4 the numerical solutions of problem (8) which is generated by the FHqOTM of order* 2γ *and compare the results with the exact solution, respectively.*

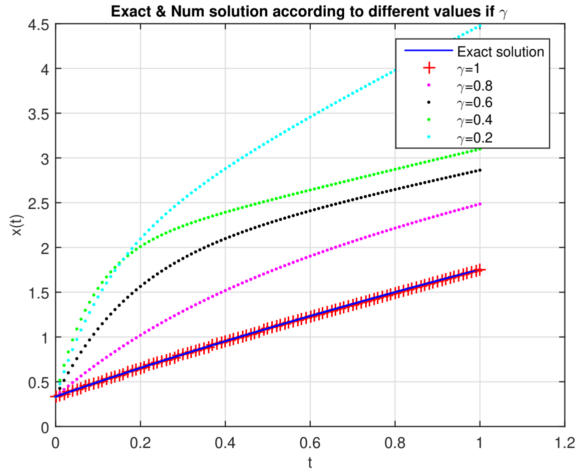

Fig. 3: *Comparisons between Exact & Numerical solutions for different values of* γ *for* $q = 0.99$.

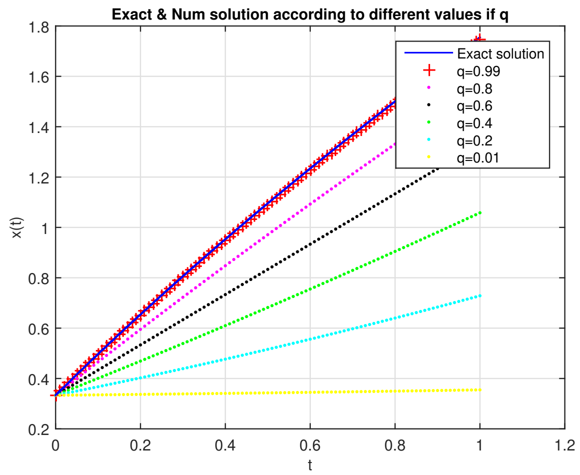

Fig. 4: *Comparisons between Exact & Numerical solutions for different values of* q *for* $\gamma = 1$.

Example 2 *Consider the following initial value problem of fractional q-difference equations:*

$$\begin{cases} {}^{C}D_q^{\gamma}x(t) = x(t) + t^3, 0 < \gamma \leq 1, t \in [0,1], \\ x(0) = 1. \end{cases} \tag{12}$$

The following is the exact solution to the problem (12) for $\gamma = 1$*:*

$$x(t) = 7e^t - t^3 - 3t^2 - 6t - 6.$$

Let $n = 100$ *in this case* $h = 0.01$*. Let's suppose:*

$$f(t, x(t)) = x(t) + t^3.$$

Next, we want to use the FHOqTM of orders 2γ and 4γ, respectively. To accomplish this, we calculate:

$$
\begin{aligned}
{}^{C}D_q^{\gamma} f(t,x(t)) &= x(t)+t^3+\frac{\Gamma_q(4)}{\Gamma_q(4-\gamma)}t^{(3-\gamma)},\\
{}^{C}D_q^{2\gamma} f(t,x(t)) &= x(t)+t^3+\frac{\Gamma_q(4)}{\Gamma_q(4-\gamma)}t^{(3-\gamma)}\\
&\quad+\frac{\Gamma_q(4)}{\Gamma_q(4-2\gamma)}t^{(3-2\gamma)},
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{C}D_q^{3\gamma} f(t,x(t)) &= x(t)+t^3+\frac{\Gamma_q(4)}{\Gamma_q(4-\gamma)}t^{(3-\gamma)}\\
&\quad+\frac{\Gamma_q(4)}{\Gamma_q(4-2\gamma)}t^{(3-2\gamma)}\\
&\quad+\frac{\Gamma_q(4)}{\Gamma_q(4-3\gamma)}t^{(3-3\gamma)}.
\end{aligned}
$$

Consequently, we can use equation (7) to find $T(t_i,v_i)$ as:

$$
\begin{aligned}
T(t_i,v_i) &= \frac{1}{\Gamma(\gamma+1)}(v_i+t_i^3)+\frac{h^{\gamma}}{\Gamma(2\gamma+1)}\Bigg(v_i+t_i^3\\
&\quad+\frac{\Gamma_q(4)}{\Gamma_q(4-\gamma)}t_i^{(3-\gamma)}\Bigg)+\frac{h^{2\gamma}}{\Gamma(3\gamma+1)}\Bigg(v_i+t_i^3\\
&\quad+\frac{\Gamma_q(4)}{\Gamma_q(4-\gamma)}t_i^{(3-\gamma)}+\frac{\Gamma_q(4)}{\Gamma_q(4-2\gamma)}t_i^{(3-2\gamma)}\Bigg)\\
&\quad+\frac{h^{3\gamma}}{\Gamma(4\gamma+1)}\Bigg(v_i+t_i^3+\frac{\Gamma_q(4)}{\Gamma_q(4-\gamma)}t_i^{(3-\gamma)}\\
&\quad+\frac{\Gamma_q(4)}{\Gamma_q(4-2\gamma)}t_i^{(3-2\gamma)}+\frac{\Gamma_q(4)}{\Gamma_q(4-3\gamma)}t_i^{(3-3\gamma)}\Bigg),
\end{aligned}
\tag{13}
$$

where v_i denote estimates for $x(t_i)$, such that $i=0,1,2,\dots,99$.
Since $t_i=t_0+ih=0.01i$ for $i=0,1,2,\dots,99$, equation (13) can be rewritten as follows:

$$
\begin{aligned}
T(t_i,v_i) &= \frac{1}{\Gamma(\gamma+1)}(v_i+(0.01i)^3)+\frac{(0.01)^{\gamma}}{\Gamma(2\gamma+1)}\\
&\quad\times\Bigg(v_i+(0.01i)^3+\frac{\Gamma_q(4)}{\Gamma_q(4-\gamma)}(0.01i)^{(3-\gamma)}\Bigg)\\
&\quad+\frac{(0.01)^{2\gamma}}{\Gamma(3\gamma+1)}\Bigg(v_i+(0.01i)^3+\frac{\Gamma_q(4)}{\Gamma_q(4-\gamma)}\\
&\quad\times(0.01i)^{(3-\gamma)}+\frac{\Gamma_q(4)}{\Gamma_q(4-2\gamma)}(0.01i)^{(3-2\gamma)}\Bigg)\\
&\quad+\frac{(0.01)^{3\gamma}}{\Gamma(4\gamma+1)}\Bigg(v_i+(0.01i)^3+\frac{\Gamma_q(4)}{\Gamma_q(4-\gamma)}\\
&\quad\times(0.01i)^{(3-\gamma)}+\frac{\Gamma_q(4)}{\Gamma_q(4-2\gamma)}(0.01i)^{(3-2\gamma)}\\
&\quad+\frac{\Gamma_q(4)}{\Gamma_q(4-3\gamma)}(0.01i)^{(3-3\gamma)}\Bigg),
\end{aligned}
\tag{14}
$$

Thus, by using formula (14), we can determine the 2γ and 4γ order of the FHOqTM respectively as:

$$
\begin{cases}
v_0=1,\\
v_{i+1}=v_i+(0.01)^{\gamma}\Big[\frac{1}{\Gamma(\gamma+1)}(v_i+(0.01i)^3)+\frac{(0.01)^{\gamma}}{\Gamma(2\gamma+1)}\\
\qquad\times\Big(v_i+(0.01i)^3+\frac{\Gamma_q(4)}{\Gamma_q(4-\gamma)}(0.01i)^{(3-\gamma)}\Big)\Big].
\end{cases}
\tag{15}
$$

And

$$
\begin{cases}
v_0=1,\\
v_{i+1}=v_i+(0.01)^{\gamma}\Big[\frac{1}{\Gamma(\gamma+1)}(v_i+(0.01i)^3)+\frac{(0.01)^{\gamma}}{\Gamma(2\gamma+1)}\\
\quad\times\Big(v_i+(0.01i)^3+\frac{\Gamma_q(4)}{\Gamma_q(4-\gamma)}(0.01i)^{(3-\gamma)}\Big)\\
\quad+\frac{(0.01)^{2\gamma}}{\Gamma(3\gamma+1)}\Big(v_i+(0.01i)^3+\frac{\Gamma_q(4)}{\Gamma_q(4-\gamma)}(0.01i)^{(3-\gamma)}\\
\quad+\frac{\Gamma_q(4)}{\Gamma_q(4-2\gamma)}(0.01i)^{(3-2\gamma)}\Big)+\frac{(0.01)^{3\gamma}}{\Gamma(4\gamma+1)}\\
\quad\times\Big(v_i+(0.01i)^3+\frac{\Gamma_q(4)}{\Gamma_q(4-\gamma)}(0.01i)^{(3-\gamma)}\\
\quad+\frac{\Gamma_q(4)}{\Gamma_q(4-2\gamma)}(0.01i)^{(3-2\gamma)}+\frac{\Gamma_q(4)}{\Gamma_q(4-3\gamma)}(0.01i)^{(3-3\gamma)}\Big)\Big].
\end{cases}
\tag{16}
$$

for $i=0,1,2,\dots,99$.
Looking at the formulas (15)-(16) and by using FHOqTM of order 2γ and 4γ respectively, we can simulate approximate solutions with exact solution to the problem (12) for $q=0.99$ and $\gamma=1$, it is shown in Fig. 5 and Fig. 7, respectively. Fig. 6 and Fig. 8 represent the absolute errors of order 2γ and 4γ, respectively.

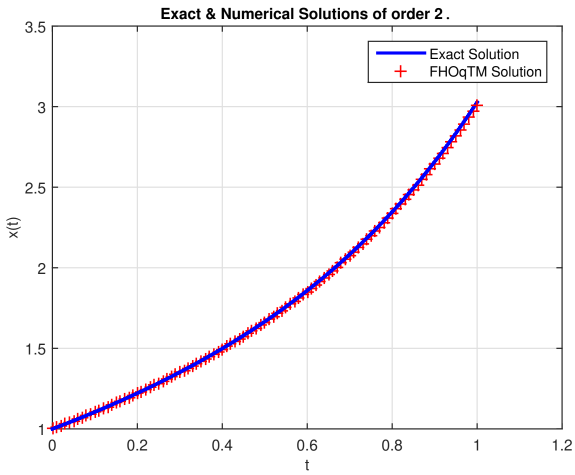

Fig. 5: *Exact & Numerical solutions of order 2γ for $q=0.99$ and $\gamma=1$.*

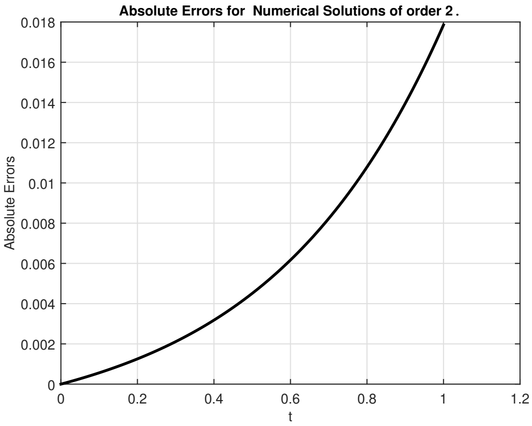

Fig. 6: *Absolute Errors of order* 2γ *for* $q = 0.99$ *and* $\gamma = 1$.

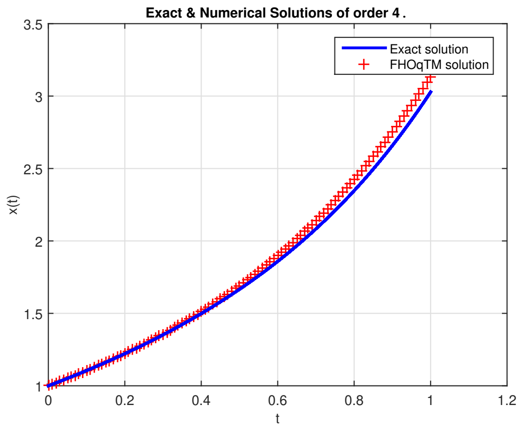

Fig. 7: *Exact & Numerical solutions of order* 4γ *for* $q = 0.99$ *and* $\gamma = 1$.

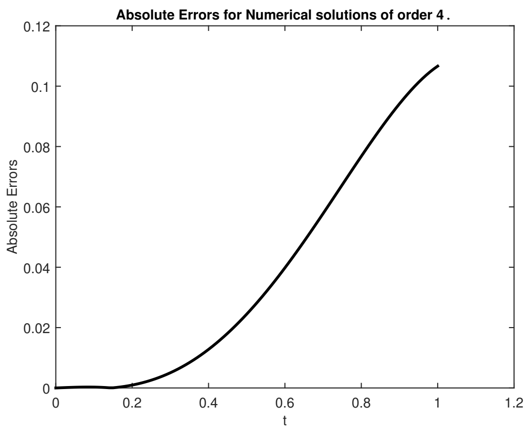

Fig. 8: *Absolute Errors of order* 4γ *for* $q = 0.99$ *and* $\gamma = 1$.

By choosing various values of γ *for* $q = 0.99$*, and different values of* q *for* $\gamma = 1$*, respectively. We plot in Fig.9 and Fig.10 the numerical solutions of problem (12) which is generated by the FHqOTM of order* 4γ *and compare the obtained results with the exact solution, respectively.*

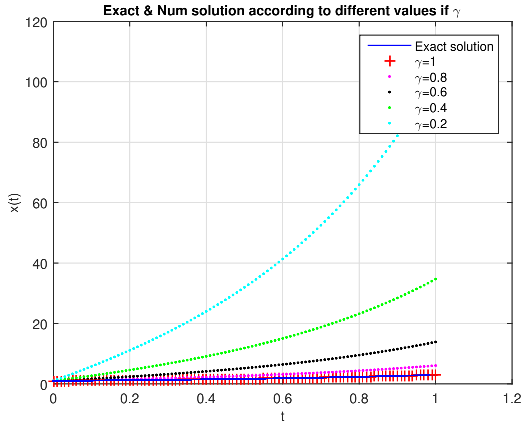

Fig. 9: *Comparisons between Exact & Numerical solutions for various values of* γ *for* $q = 0.99$.

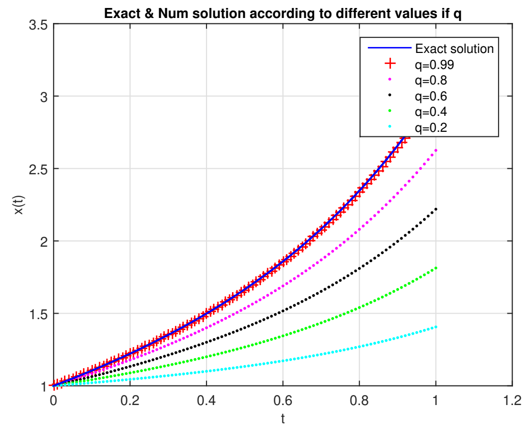

Fig. 10: *Comparisons between Exact & Numerical solutions for various values of* q *for* $\gamma = 1$.

CONCLUSION

In this research, we establish a novel fractional approach to solve the initial value problem of fractional q-difference equations involving the Caputo's q-derivative, known as the Fractional Higher-Order q-Taylor Method (FHOqTM). This method is based on the generalised q-Taylor's theorem, and we have estimate the local truncation error that FHOqTM produces. In order to show the efficacy of our results, we presented the numerical applications and performed several comparisons between the exact solutions and the approximate solution generated by the FHqOTM. The suggested approach has proven to be a successful numerical method for resolving the initial value problems of fractional q-difference equations.

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