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STEREOLOGY WITH CYLINDER PROBES

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ABSTRACT

Intersection formulae of Crofton type for general geometric probes are well known in integral geometry. For the special case of cylinders with non necessarily convex directrix, however, no equivalent formula seems to exist in the literature. We derive this formula resorting to motion invariant probability elements associated with test systems, instead of using a traditional approach. Because cylinders are seldom used as probes in streological practice, however, this note is mainly of a theoretical nature.

Keywords: Cylinders, integral geometry, motion invariant measures, ratio design, stereology, test systems.

INTRODUCTION

The fundamental equations of stereology - see for instance Miles (1972), Baddeley and Jensen (2005), (Section 2.2.3), or Cruz-Orive (2017) for their history, are based on intersections between a target set and a geometric probe. The latter is usually an $r-$plane, or a bounded portion of it. With rare exceptions (e.g. Horgan *et al.* (1993)) cylinder probes are seldom used in stereology - therefore, the present note is mainly theoretically oriented.

The classical stereological equations are usually ratios of motion invariant measures. For instance, the ratio B_A of the total planar curve length determined by a motion invariant test plane in the boundary of a compact three dimensional set, divided by the total planar section area determined in the set, is equal to $(\pi/4)S_V$, where S_V is the surface to volume ratio of the set. The identity is the result of dividing side by side two integral identities which belong to the family of Crofton intersection formulae of integral geometry. Such ratio identities hold formally unchanged for probes other than $r-$planes, notably cylinders. The Crofton integrals in the numerator and the denominator of a ratio, however, do in general depend on probe shape.

To fix ideas, consider a cylindrical surface $Z_2 \subset \mathbb{R}^3$ whose generator $L_{1[0]}$ is a straight line and its directrix, namely its cross section by a plane $L_{2[0]}$ through the origin, perpendicular to the generator, is a piecewise smooth, simple closed curve $Z_1 \subset L_{2[0]}$ of perimeter length b, see Fig. 1. The object is a compact set $Y \subset \mathbb{R}^3$ of surface area S and volume V. The motion invariant density of the cylinder is the kinematic density:

$$\mathrm{d}Z_2 = \mathrm{d}x\,\mathrm{d}u\,\mathrm{d}\tau, \quad x \in \mathbb{R}^2, u \in \mathbb{S}^2, \tau \in \mathbb{S}, \tag{1}$$

where x is an associated point (AP) of the cross section Z_1, (namely a point rigidly attached to Z_1 according to a fixed rule), whereas u is a unit vector on the unit sphere $\mathbb{S}^2$ giving the direction of the generator, and τ is a rotation around the generator. The pertinent Crofton intersection formulae read as follows,

$$\int B(\partial Y \cap Z_2)\,\mathrm{d}Z_2 = 2\pi^3 bS, \tag{2}$$

$$\int A(Y \cap Z_2)\,\mathrm{d}Z_2 = 8\pi^2 bV, \tag{3}$$

and side by side division yields the aforementioned stereological equation $S_V = (4/\pi)B_A$. Analogous formulae arise for a solid cylinder $Z_3 \subset \mathbb{R}^3$ whose cross section $Z_2 \subset L_{2[0]}$ is a domain of area a and perimeter length b.

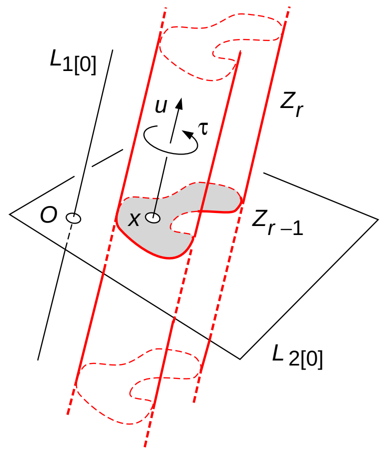

Fig. 1. *Sketch of a cylinder $Z_r \subset \mathbb{R}^3$ with generator $L_{1[0]}$ and directrix $Z_{r-1} \subset L_{2[0]}$. For $r = 1,2,3$ the cylinder is a straight line, a cylindrical surface, or a solid cylinder, respectively. The remaining symbols are defined in the text.*

While a Crofton intersection formula for general manifolds is well known, see for instance Santaló (1976), Eq. (15.20), we have not found an analogous formula for general cylinders in the literature. Schneider and Weil (2008) consider cylinders with convex directrix. Particular cases such as Eq. 2 and Eq. 3, among others, can be found in Santaló (1936), Eq. (115)-(118), who, in his Eq. (104), uses an invariant density dZ analogous to Eq. 1 for an unoriented cylinder Z, namely for $u \in \mathbb{S}_+^2$, whereby the relevant results are halved. Rey-Pastor and Santaló (1951) use the correct invariant density (Eq. (37.1)) and derive particular cases, namely Eq. (37.20), which is the same as Eq. 2 above, and Eq. (37.17) for the integral of the number of intersections between a cylinder surface and a curve in $\mathbb{R}^3$, see Eq. 35 below. In either of the preceding two publications the main emphasis was more on hitting measures (e.g. the measure of the number of cylinders hitting a compact set) than on intersection measures. This probably obeyed to the popularity of geometric inequalities (notably Minkowski's) over the first half of the 20th century. These preferences (at least as far as cylinders is concerned) were inherited by the books of Hadwiger (1957) and Santaló (1976). In p. 280, the latter book just reproduces the two particular formulae from Rey-Pastor and Santaló (1951) cited above.

Rey-Pastor and Santaló (1951) derive their Eq. (37.17) and Eq. (37.20) directly using *ad hoc* arguments which, being ingenious and elegant, lack a pattern that can be easily generalized. Here we derive a general Crofton intersection formula for cylinders (see Eq. 22 below) using invariant probability measures associated with test systems, instead of the traditional tools of integral geometry.

To make the note self contained, the main prerequisites are given next; the formula, and its proof, are given in the *Results* section.

DEFINITIONS AND NOTATION

RELEVANT GEOMETRIC OBJECTS

A cylinder $Z_r^n \subset \mathbb{R}^n$ of dimension $r \in \{1,2,\ldots,n\}$ is defined as the set product

$$Z_r^n = L_{p[0]}^n \times Z_{r-p}^{n-p}, \quad p \in \{1,2,\ldots,r\}, \tag{4}$$

where $L_{p[0]}^n$ is a linear subspace of dimension p through a fixed origin $O \in \mathbb{R}^n$, whereas Z_{r-p}^{n-p} is a compact submanifold of dimension $r-p$ of class C^2, contained in the linear subspace $L_{n-p[0]}^n$ perpendicular to $L_{p[0]}^n$.

Thus, Z_{r-p}^{n-p} may be visualized as:

a) $Z_r^n \cap L_{n-p[0]}^n$, the $(r-p)-$dimensional cross-section of Z_r^n determined by the $(n-p)-$plane $L_{n-p[0]}^n$.

b) The $(r-p)-$dimensional orthogonal projection of the cylinder Z_r^n onto $L_{n-p[0]}^n$.

The linear subspace $L_{p[0]}^n$ is the generator, whereas the subset Z_{r-p}^{n-p} is the directrix of the cylinder Z_r^n.

Examples for $n = 3$ and $p = 1$.

• If $r = 1$, then the cylinder Z_1^3 is a straight line L_1^3 normal to a given plane $L_{2[0]}^3$.

• If $r = 2$, then Z_2^3 is a cylindrical surface whose cross section is a bounded curve Z_1^2 contained in a plane $L_{2[0]}^3$ normal to the straight line generator $L_{1[0]}^3$.

• If $r = 3$, then Z_3^3 is a solid cylinder whose cross section is a domain Z_2^2 of dimension 2 contained in $L_{2[0]}^3$.

An $n-$box $J_0^n \subset \mathbb{R}^n$ is defined as

$$J_0^n = [0,a_1)\times[0,a_2)\times\cdots\times[0,a_n), \tag{5}$$

where $a_1, a_2, \ldots, a_n$ are finite, positive real numbers.

A bounded cylinder $T_r^n \subset \mathbb{R}^n$ is a compact set defined as:

$$T_r^n = Z_{r-p}^{n-p} \times J_0^p. \tag{6}$$

Example. If $p = 1$, then T_r^n is a bounded right cylinder of base Z_{r-1}^{n-1} and finite height $a_1 > 0$.

MOTION INVARIANT DENSITIES

The material in this section, given to make the note self contained, is well known - for general reference see Santaló (1976).

A non-oriented linear subspace $L_{p[0]}^n$ submitted to a rotation from the group $G_{p,n-p}$ of rotations about a fixed point in $\mathbb{R}^n$, called the Grassmann manifold, has a rotation invariant density denoted by

$$\mathrm{d}L_{p[0]}^n = \mathrm{d}L_{n-p[0]}^n. \tag{7}$$

It can be shown that

$$\int_{G_{p,n-p}} \mathrm{d}L_{p[0]}^n = \frac{O_{n-1}O_{n-2}\cdots O_{n-p}}{O_{p-1}O_{p-2}\cdots O_0}, \tag{8}$$

where

$$O_k = \frac{2\pi^{(k+1)/2}}{\Gamma((k+1)/2)}, \quad k = 0,1,\ldots,n, \tag{9}$$

denotes the surface area of the $k-$dimensional unit sphere $\mathbb{S}^k$ ($O_0 = 2$, $O_1 = 2\pi$, $O_{k+2} = 2\pi O_k/(k+1)$).

As given below, the motion invariant density for cylinders involves an oriented $p-$subspace $L^n_{p[0]}$, which we denote by $L^*_{p[0]}$. Consequently, the measure given in Eq. 8 must be multiplied by $O_0 = 2$.

For a compact set $T^n_r \subset \mathbb{R}^n$, not necessarily a bounded cylinder, the motion invariant density is the kinematic density, namely,

$$\mathrm{d}T^n_r = \mathrm{d}x_n\,\mathrm{d}u_n, \quad x_n \in \mathbb{R}^n,\ u_n \in G_{n[0]}, \tag{10}$$

where x_n is an AP fixed in T^n_r, whereas u_n is an element of the special group of rotations $G_{n[0]}$, isomorphic to $SO(n)$, about a fixed point in $\mathbb{R}^n$. It can be shown that

$$\int_{G_{n[0]}} \mathrm{d}u_n = O_{n-1}O_{n-2}\ldots O_1. \tag{11}$$

Example. For $n = 3$ we have $u_3 = (u_2, u_1)$, where $u_2 \equiv (\phi, \theta) \in \mathbb{S}^2$ is a unit vector of spherical polar coordinates (ϕ, θ) whereas $u_1 \equiv \tau \in \mathbb{S}^1$ is a rotation about u_2. Thus,

$$\int_{G_{3[0]}} \mathrm{d}u_3 = \int_{\mathbb{S}^2} \mathrm{d}u_2 \int_{\mathbb{S}^1} \mathrm{d}u_1 = 4\pi\,2\pi = 8\pi^2. \tag{12}$$

For a cylinder Z^n_r of cross section Z^{n-p}_{r-p}, where $1 \le p \le r \le n$, the motion invariant density is

$$\mathrm{d}Z^n_r = \mathrm{d}Z^{n-p}_{r-p}\ \mathrm{d}L^*_{p[0]}, \tag{13}$$

(Santaló, 1976, Eq. (15.76)), where $\mathrm{d}Z^{n-p}_{r-p}$ is the kinematic density in $L_{n-p[0]}$, namely,

$$\mathrm{d}Z^{n-p}_{r-p} = \mathrm{d}x_{n-p}\ \mathrm{d}u_{n-p}. \tag{14}$$

Substitution into the right hand side of Eq. 13 yields,

$$\mathrm{d}Z^n_r = \mathrm{d}x_{n-p}\,\mathrm{d}L^*_{p[0]}\,\mathrm{d}u_{n-p}, \tag{15}$$

where $x_{n-p} \in L^n_{n-p[0]}$, $L^*_{p[0]} \in G^*_{p,n-p}$, $u_{n-p} \in G_{n-p[0]}$ (this group is isomorphic to the group of rotations about $L^*_{p[0]}$). From Eq. 8 and Eq. 11 we get,

$$\int_{G^*_{p,n-p}} \mathrm{d}L^*_{p[0]} \int_{G_{n-p[0]}} \mathrm{d}u_{n-p} = O_{n-1}\ldots O_p. \tag{16}$$

Example. For a cylinder Z^3_r, $r = 2, 3$, $p = 1$, the motion invariant density given by Eq. 15 reduces to Eq. 1.

TEST SYSTEMS OF CYLINDERS

Consider a test system $\Lambda_T \subset \mathbb{R}^n$ whose fundamental tile is an $n-$box J^n_0, whereas the fundamental probe is a bounded cylinder $T^n_r \subset J^n_0$ given by Eq. 6 with its AP at the origin O, see Fig. 2. For details pertaining to the construction of a test system see Santaló (1976) under the term "lattice of figures".

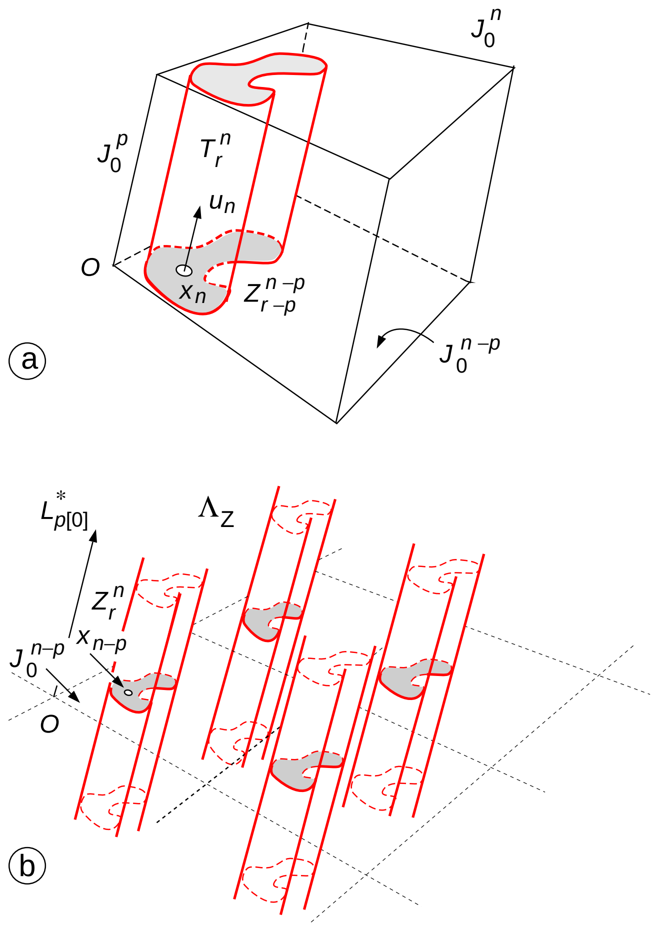

Fig. 2. *(a) Fundamental tile (n−box) J^n_0 of a test system Λ_T. The fundamental probe $T^n_r \subset J^n_0$ is a bounded cylinder given by Eq. 6. (b) A portion of the test system Λ_T generated by T^n_r, see (a), which is congruent with a test system Λ_Z of cylinders.*

The motion invariant probability element adopted for Λ_T, for which we use the shorthand notation $\mathbb{P}(\mathrm{d}\Lambda_T)$, is the normalitzed version of the kinematic density $\mathrm{d}T^n_r$, as follows. For $x_n \in J^n_0$, $u_n \in G_{n[0]}$,

$$\mathbb{P}(\mathrm{d}\Lambda_T) \equiv \mathbb{P}(\mathrm{d}x_n,\ \mathrm{d}u_n) = \frac{\mathrm{d}x_n}{\nu_n(J^n_0)}\,\frac{\mathrm{d}u_n}{\int \mathrm{d}u_n}, \tag{17}$$

where $\nu_d(\cdot)$ denotes $d-$Hausdorff measure. The integral is given by Eq. 11. For details see Cruz-Orive (2002), or Appendix A in Voss and Cruz-Orive (2009).

Because $T^n_r = Z^{n-p}_{r-p} \times J^p_0$, $J^n_0 = J^{n-p}_0 \times J^p_0$, and $Z^{n-p}_{r-p} \subset J^{n-p}_0$, any stack of probes of Λ_T is in fact an infinite cylinder Z_r^n. Thus, Λ_T coincides with a test system $\Lambda_Z \subset \mathbb{R}^n$ of cylinders congruent with Z_r^n, whose fundamental tile is $J_0^{n-p} \subset L_{n-p[0]}^n$, whereas the fundamental probe is the orthogonal projection Z_{r-p}^{n-p} of Z_r^n onto $L_{n-p[0]}^n$. The motion invariant probability element corresponding to Λ_Z is the normalized density given by Eq. 15. Thus, for $x_{n-p} \in J_0^{n-p}$, $L_{p[0]}^* \in G_{p,n-p}^*$, $u_{n-p} \in G_{n-p[0]}$,

$$\begin{aligned} \mathbb{P}(\mathrm{d}\Lambda_Z) &\equiv \mathbb{P}(\mathrm{d}x_{n-p},\, \mathrm{d}L_{p[0]}^*,\, \mathrm{d}u_{n-p}) \\ &= \frac{\mathrm{d}x_{n-p}}{\nu_{n-p}(J_0^{n-p})} \frac{\mathrm{d}L_{p[0]}^*\, \mathrm{d}u_{n-p}}{\int \mathrm{d}L_{p[0]}^* \mathrm{d}u_{n-p}}, \end{aligned} \quad (18)$$

where the integral involving the orientation variables is given by Eq. 16.

From the preceding considerations it follows that the test system Λ_Z equipped with the probability element given by Eq. 18, has identical statistical properties as the test system Λ_T equipped with the probability element given by Eq. 17. In particular, for a compact submanifold $Y_q \subset \mathbb{R}^n$ of dimension $q \in \{0,1,\ldots,n\}$, with $q+r \geq n$, the following identity holds,

$$\mathbb{E}\nu_{q+r-n}(Y_q \cap \Lambda_T) = \mathbb{E}\nu_{q+r-n}(Y_q \cap \Lambda_Z), \quad (19)$$

the expectations being with respect to the corresponding motion invariant probability elements.

CROFTON INTERSECTION FORMULA FOR BOUNDED PROBES

For the compact submanifold $Y_q \subset \mathbb{R}^n$ just considered, hit by a compact probe T_r^n equipped with the kinematic density $\mathrm{d}T_r^n$ given by Eq. 10, the following identity holds,

$$\begin{aligned} &\int_{\mathbb{R}^n \times G_{n[0]}} \nu_{q+r-n}(Y_q \cap T_r^n)\, \mathrm{d}T_r^n \\ &\quad = c(q,r,n)\nu_q(Y_q)\, \nu_r(T_r^n), \end{aligned} \quad (20)$$

(Santaló (1976), Eq. (15.20)), where

$$c(q,r,n) = \frac{O_n O_{n-1} \cdots O_1 O_{q+r-n}}{O_q O_r}. \quad (21)$$

RESULTS

CROFTON INTERSECTION FORMULA FOR CYLINDERS

The main purpose of this note is to prove the following identity.

Proposition

$$\begin{aligned} &\int_{\mathbb{R}^{n-p} \times G_{p,n-p}^* \times G_{n-p[0]}} \nu_{q+r-n}(Y_q \cap Z_r^n)\, \mathrm{d}Z_r^n \\ &\quad = c_Z(q,r,n,p)\nu_{r-p}(Z_{r-p}^{n-p})\, \nu_q(Y_q) \end{aligned} \quad (22)$$

where

$$c_Z(q,r,n,p) = \frac{O_n O_{n-1} \cdots O_p O_{q+r-n}}{O_q O_r}. \quad (23)$$

Proof

Set $\nu \equiv \nu_{q+r-n}$, $Y \equiv Y_q$, $T \equiv T_r^n$, $Z \equiv Z_r^n$, $Z' \equiv Z_{r-p}^{n-p}$ and $c \equiv c(q,r,n)$, for short. In addition, the domains of integration $G_{n[0]}$ for u_n, $G_{p,n-p}^*$ for $L_{p[0]}^*$, and $G_{n-p[0]}$ for u_{n-p}, will be omitted in the sequel.

The proof is based on Eq. 19, whose left hand side becomes,

$$\begin{aligned} &\mathbb{E}\nu(Y \cap \Lambda_T) \\ &= \int_{x_n \in J_0^n} \nu(Y \cap \Lambda_T)\, \mathbb{P}(\mathrm{d}\Lambda_T) \\ &= \frac{1}{\nu_n(J_0^n) \int \mathrm{d}u_n} \int_{x_n \in J_0^n} \nu(Y \cap \Lambda_T)\, \mathrm{d}x_n\, \mathrm{d}u_n \\ &= \frac{1}{\nu_n(J_0^n) \int \mathrm{d}u_n} \int_{x_n \in \mathbb{R}^n} \nu(Y \cap T)\, \mathrm{d}T, \end{aligned} \quad (24)$$

where the last identity follows from Santaló's identity for test systems, see Santaló (1976), chapter 8, or Cruz-Orive (2002). In combination with Eq. 20 we obtain,

$$\mathbb{E}\nu(Y \cap \Lambda_T) = \frac{c\nu_q(Y)\nu_r(T)}{\nu_n(J_0^n) \int \mathrm{d}u_n}. \quad (25)$$

Analogously,

$$\begin{aligned} &\mathbb{E}\nu(Y \cap \Lambda_Z) \\ &= \int_{x_{n-p} \in J_0^{n-p}} \nu(Y \cap \Lambda_Z)\, \mathbb{P}(\mathrm{d}\Lambda_Z) \\ &= \frac{\int_{x_{n-p} \in \mathbb{R}^{n-p}} \nu(Y \cap Z)\, \mathrm{d}Z}{\nu_{n-p}(J_0^{n-p}) \int dL_{p[0]}^* \mathrm{d}u_{n-p}}. \end{aligned} \quad (26)$$

Finally, bearing Eq. 11 and Eq. 16 in mind, applying Eq. 19, and using the following identities,

$$\nu_n(J_0^n) = \nu_{n-p}(J_0^{n-p})\, \nu_p(J_0^p), \quad (27)$$

$$\nu_r(T) = \nu_{r-p}(Z')\, \nu_p(J_0^p), \quad (28)$$

we obtain,

$$\begin{aligned} &\int_{x_{n-p} \in \mathbb{R}^{n-p}} \nu(Y \cap Z)\, \mathrm{d}Z \\ &= \frac{O_n O_{n-1} \cdots O_p O_{q+r-n}}{O_q O_r} \nu_{r-p}(Z')\nu_q(Y). \end{aligned} \quad (29)$$

which is the identity given by Eq. 22, thus completing the proof of the proposition. □

SPECIAL CASES FOR $n = 2,3$ AND $p = 1$

A cylinder $Z_2^2 \subset \mathbb{R}^2$ is a solid stripe of thickness $t > 0$, say, in the plane. Its boundary $\partial Z_2^2 \equiv Z_1^2$ is the union of two parallel straight lines a distance t apart.

Consider a compact set $Y_2 \subset \mathbb{R}^2$ of area $A > 0$ and piecewise smooth boundary $\partial Y_2 \equiv Y_1$ of length $B > 0$. Application of Eq. 22 with $r = 1$ yields:

$$\int \nu_0(Y_1 \cap Z_1^2)\, \mathrm{d}Z_1^2 = \frac{O_2 O_1 O_0}{O_1 O_1} \nu_0(Z_0^1)\, B = 8B, \quad (30)$$

because the projection Z_0^1 of Z_1^2 onto an axis normal to the stripe is the union of two points a distance t apart - hence $\nu_0(Z_0^1) = 2$. As a cross-check, note that

$$\begin{aligned} \int \nu_0(Y_1 \cap Z_1^2)\, \mathrm{d}Z_1^2 &= 2 \int I(Y_1 \cap L_1^*)\, \mathrm{d}L_1^* \\ &= 4 \int I(Y_1 \cap L_1^2)\, \mathrm{d}L_1^2 = 8B, \end{aligned} \quad (31)$$

where L_1^2 is a straight line with motion invariant density $\mathrm{d}L_1^2$, and $I(\cdot)$ denotes number of intersections - see Cruz-Orive (2017), Eq. (1), for references.

On the other hand

$$\int \nu_2(Y_2 \cap Z_2^2)\, \mathrm{d}Z_2^2 = \frac{O_2 O_1 O_2}{O_2 O_2} \nu_1(Z_1^1)\, A = 2\pi t A, \quad (32)$$

which is twice the value obtained in the classical manner (see Eq. (5.16) of Santaló (1976)), because we consider oriented stripes. Note that the projection Z_1^1 of the stripe Z_2^2 onto an axis normal to the stripe is a segment of length t - hence $\nu_1(Z_1^1) = t$.

Further

$$\int \nu_1(Y_2 \cap Z_1^2)\, \mathrm{d}Z_1^2 = \frac{O_2 O_1 O_1}{O_2 O_1} \nu_0(Z_0^1)\, A = 4\pi A, \quad (33)$$

which is equivalent to

$$\int \nu_1(Y_2 \cap Z_1^2)\, \mathrm{d}Z_1^2 = 4 \int L(Y_2 \cap L_1^2)\, \mathrm{d}L_1^2 = 4\pi A, \quad (34)$$

where $L(\cdot)$ denotes intercept length, and the first integral pertains to the two oriented straight lines constituting Z_1^2, yielding $2\pi A$ each.

For $n = 3$ and $r = q = 2$, Eq. 22 yields $c_Z(2,2,3,1) = O_3 O_2 O_1 O_1/(O_2 O_2) = 2\pi^3$, and we obtain Eq. 2.

For $n = 3$, $q = 3$, and $r = 2$ Eq. 22 yields $c_Z(3,2,3,1) = O_3 O_2 O_1 O_2/(O_3 O_2) = 8\pi^2$ and we obtain Eq. 3.

Finally, for $n = 3$, $r = 2$, and $q = 1$, namely for $Z_2^3 \subset \mathbb{R}^3$ hitting a curve $Y_1 \subset \mathbb{R}^3$ of length L, we have $c_Z(1,2,3,1) = O_3 O_0 = 4\pi^2$ and,

$$\int \nu_0(Y_1 \cap Z_2^3)\, \mathrm{d}Z_2^3 = 4\pi^2 bL, \quad (35)$$

see Santaló (1976), p. 280.

STEREOLOGICAL EQUATIONS FOR TEST SYSTEMS OF CYLINDERS

Substitution of Eq. 29 into the right hand side of Eq. 26, yields the Hausdorff measure $\nu_q(Y_q)$ of a compact submanifold $Y_q \subset \mathbb{R}^n$ in terms of the measure of its intersection with a test system Λ_{Z_r} of cylinders of dimension r, namely,

$$\nu_q(Y_q) = \frac{O_q O_r}{O_n O_{q+r-n}} \frac{\nu_{n-p}(J_0^{n-p})}{\nu_{r-p}(Z_{r-p}^{n-p})} \cdot \mathbb{E}\nu_{q+r-n}(Y_q \cap \Lambda_{Z_r}). \quad (36)$$

The numerical constant in the right hand side of the preceding identity is the same as that arising for $r-$probes in general, see for instance Voss and Cruz-Orive (2009), Eq. (A28), (A32).

In stereology, $\nu_q(Y_q)$ is often estimated via the ratio design, which is based on the identity,

$$\nu_q(Y_q) = \nu_n(X_n) \cdot R_{q,n}, \quad (37)$$

where $X_n \supset Y_q$ is a reference submanifold containing Y_q, whose volume is estimated separately (e.g. by the Cavalieri design). Ratio designs were studied in some detail by Cruz-Orive (1980) and Cruz-Orive and Weibel (1981), see also Baddeley and Jensen (2005). Thus, it only remains to estimate the ratio $R_{q,n} \equiv \nu_q(Y_q)/\nu_n(X_n)$ via the identity,

$$R_{q,n} = \frac{O_q O_r}{O_n O_{q+r-n}} \cdot \frac{\mathbb{E}\nu_{q+r-n}(Y_q \cap \Lambda_{Z_r})}{\mathbb{E}\nu_r(X_n \cap \Lambda_{Z_r})}. \quad (38)$$

Provided that the same test system Λ_{Z_r} is used in the numerator and denominator, the right hand side of the preceding identity does not involve any properties of Λ_{Z_r} itself - hence the relative popularity of ratios. Thus, with the usual conditions Eq. 38 holds for any $r-$dimensional test system; it was already obtained by Miles (1972), Eq. (2.16), and it encapsulates the classical stereological equations used in practice, see also Cruz-Orive (2002), Eq. (6.19).

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