TRANSFORMATION OF A GRID OF QUADRATS TO COPE WITH PERSPECTIVE ARTIFACTS

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ABSTRACT
The direct superimposition of a standard test grid of congruent quadrats onto an image bearing a population of particles exhibiting perspective artifacts, tends to increase the variance of the population size estimator, because the quadrat contents become unbalanced. If the quadrats are transformed according to the same projection mechanism affecting the particles, however, then the variance is restored into moderate values. Our purpose was to provide exact, easily programmable equations for the relevant transform.

Keywords: cylindrical projection, gigapixel image, particle number, quadrat grid, sinusoidal curve, transformed quadrat grid.

INTRODUCTION
This note was motivated by the problem of estimating the finite size of a fixed and bounded population of objects in a reference plane. An object is regarded as a 'particle', namely a compact and connected set separated from other particles. In practice the particles (people, animals, or whatever distinguishable objects) are observable in a reference projection plane, henceforth called the horizontal plane (HP), and observed in a flat image – usually a photograph – called the image plane (IP). Ideally, the IP should be parallel – at least approximately – to the HP, in order to avoid perspective artifacts. Cruz et al. (2015) proposed to superimpose on the IP a uniform random grid of quadrats, with a fundamental square tile of area $a > 0$, containing a fundamental square quadrat of area $a' \in (0, a]$, whereby the sampling period is $a/a' \geq 1$ and

$$\hat{N} = \frac{a}{a'} \cdot Q,$$

is an unbiased estimator (UE) of the population size $N$. Here $Q$ is the sample size, namely the total number of particles captured by the quadrats. For general particles, $Q$ is scored using the forbidden line rule of Gundersen (1977), widely used in stereology to cope with edge effects. The free software CountEm (https://countem.unican.es/) offers a computer assisted procedure to estimate $N$ on real images. For further details on the sampling design see Gómez et al. (2019).

For large populations (e.g., for $N \gtrsim 10,000$), panoramic ‘gigapixel’ images are nowadays obtained with the aid of special cameras which use cylindrical projections from some elevated observation point which plays the role of a centre of projection. The resulting images exhibit perspective artifacts: the observed particle density increases with the distance from the observation point. As a consequence, distant quadrats will tend to capture many more particles than proximal quadrats, this causing $\text{Var}(\hat{N})$ to become unduly large (Cruz and González-Villa, 2019). An idea proposed in the latter paper was to apply to the quadrats the same projection mechanism used for the gigapixel image. Now, for a given position of the grid the individual quadrat counts will be identical to those obtained with an ordinary grid on an IP parallel to the HP, whereby the corresponding estimation variance will be restored. Each transformed quadrat was drawn approximately by applying the pertinent point transform to each of a few landmarks in the quadrat boundary, then joining the transformed landmarks. In this note we give explicit projection formulæ to plot the transformed grid of quadrats directly on the IP. This is done by computing first the cylindrical projection of an ordinary grid from a fixed point, and then unrolling the cylinder. The edges of the transformed quadrats turn out to be sinusoidal arcs.

CYLINDRICAL PROJECTIONS

SETUP
A special camera for gigapixel images consists of a lens which can rotate around a vertical axis; at each
position, the camera collects a partial, essentially flat image, and a suitable software creates a paving of such images which can map the target object into a panoramic view of up to 360°. The result is roughly equivalent to projecting the object through a pinhole onto the internal face of a right circular cylinder. Based on this fact, the model described below involves projection of points and lines in the HP through a centre of projection onto a vertical right circular cylinder. For clarity of exposition the projections are made onto the external surface of the cylinder, but the results are equally valid for internal projections.

Fix an orthogonal reference frame $(x_1,x_2,x_3)$ in $\mathbb{R}^3$. We assume that the $N$ particles of the target population are mapped 1:1 into a bounded set of $N$ associated points $\{y_1,y_2,\ldots,y_N\}$ contained in a horizontal plane (HP) $x_3 = h$, where $h$ is a constant. The origin $o$ is adopted as the centre of projection. The HP will usually lie below $o$, hence to visualize the setup we may assume that $h < 0$, see Fig. 1a. Consider a vertical, right circular cylinder of radius $R$ with the $x$-axis as the centre of projection onto a vertical right circular cylinder. For clarity of exposition the projections are made onto the external surface of the cylinder, but the results are equally valid for internal projections.

A cylindrical projection from $o$ maps the straight line into the ellipse $C \cap L_2$, see Fig. 1c, where $L_2$ denotes the plane containing $o$ and the straight line $L_1$, namely,

$$L_2 = \{(x_1,x_2,x_3) : hx_1 \cos \phi + hx_2 \sin \phi - px_3 = 0\}.$$  

Set $\phi = \phi - \pi/2$, and let $\alpha \in [0,2\pi)$ be the polar angle measured in the horizontal reference plane $x_1x_2$ with origin at an axis which makes an angle $\phi$ with the $x_3$ axis, (namely with an axis parallel to $L_1$ through $o$). The parametric equations of the ellipse $C \cap L_2$ in terms of $\phi$ and $\alpha$ are the following,

$$\begin{align*}
    x_1 &= r \cos(\phi + \alpha), \\
    x_2 &= r \sin(\phi + \alpha), \\
    x_3 &= (hr/p) \sin \alpha.
\end{align*}$$

The preceding expression of $x_3$ is found by substituting the expressions of $x_1, x_2$ into the equation of the plane $L_2$ containing the ellipse. Note that $x_3$ does not depend on $\phi$. Thus, the parametric equations of the ellipse map $C \cap L_2$ in the IP become

$$\begin{align*}
    X_1 &= r(\phi + \alpha), \\
    X_2 &= (hr/p) \sin \alpha,
\end{align*}$$

or, in Cartesian coordinates,

$$X_2 = \frac{hr}{p} \sin \left(\frac{X_1}{r} - \phi\right),$$

which corresponds to a sinusoidal curve, as expected, see e.g. Steinhaus (1950), Ch.10, and Fig. 1d.

### CYLINDRICAL PROJECTION OF A SQUARE GRID OF QUADRATS

For convenience we define a square grid of square quadrats in the HP as follows. A stripe $L_t(p,\phi)$ of thickness $t > 0$ is the portion of the plane comprised between two parallel straight lines of normal coordinates $(p,\phi)$ and $(p + t, \phi)$, respectively. A systematic series of stripes (called Cavalieri stripes in stereology) of fixed period $T \geq t$ is

$$\{L_t(z + kT,\phi), k \in \mathbb{Z}\},$$

**CYLINDRICAL PROJECTION OF A STRAIGHT LINE**

Consider a straight line of normal coordinates $p > 0, \phi \in [0,2\pi)$ contained in the HP, namely,

$$L_1 = \{(x_1,x_2,h) : x_1 \cos \phi + x_2 \sin \phi - p = 0\}.$$
where $z \sim \text{UR}[0, T)$ is a uniform random offset in the interval $[0, T)$, see Fig. 2a. Then, the intersection between two mutually perpendicular Cavalieri stripe series with independent UR offsets $z_1, z_2$ in $[0, T)$, namely,

$$\Lambda_{t,T}(\phi) = \{L_z(z_1 + jT, \phi)\} \cap \{L_z(z_2 + kT, \phi + \pi/2), j, k \in \mathbb{Z}\}, \phi \in [0, \pi/2),$$

is a square grid of square quadrats of side length $t$ and gap length $T - t \geq 0$, see Fig. 2b.

By virtue of Eq. 9, the transform of a stripe $\{L_z(z + kT, \phi)\}$ in the IP is the portion of space comprised between two sinusoidal curves of equations

$$\begin{align*}
X_2 &= \frac{hr}{z + kT} \cdot \sin \left(\frac{X_1}{r} - \varphi\right), \\
X_2 &= \frac{hr}{z + kT + t} \cdot \sin \left(\frac{X_1}{r} - \varphi\right),
\end{align*}$$

respectively. For the transform of the perpendicular stripe $L_z(z + jT, \phi + \pi/2)$ the equations are analogous to the preceding ones with $\sin(\cdot)$ replaced with $\cos(\cdot)$.

With this information it is easy to write a software routine to plot the transform of the test system $\Lambda_{t,T}(\phi)$ in the IP, see Fig. 2b,c. Note that a change in the orientation $\phi$ of the original test system causes only a shift of the transformed test system along the horizontal axis $OX_1$ in the IP.
CRUZ-ORIVE LM ET AL: Transformation of a grid of quadrats

Fig. 2. (a) Quadrat generation as the intersection of two mutually perpendicular, uniform random and independent systems of Cavalieri stripes. (b) A portion of a square test system of square quadrats in the horizontal plane. The quadrats bear a portion of the forbidden edge (in red) to apply the corresponding unbiased counting rule. (c) Generation of the transformed test system in the image plane by means of the intersection of two families of sinusoids with $\phi = 0$ and $\phi = \pi/2$, respectively, see Eq. 12. The transformed quadrats should also bear the corresponding forbidden lines, which are portions of sinusoidal curves (see Fig. 3c).

A NOTE ON THE PERSPECTIVE EFFECT

Consider two points $y_1, y_2$ in the HP, of cylindrical coordinates $(R, \omega, h)$ and $(R + \delta, \omega, h)$ respectively. Thus, for simplicity the two points are collinear with the point $o'(0,0,h)$, with a distance $\delta > 0$ between them. By the second Eq. 4, the transforms of $y_1, y_2$ in the IP lie in a vertical line, and have ordinates $X_{21} = hr/R$ and $X_{22} = hr/(R + \delta)$, respectively. Thus, the apparent distance between the two point transforms in the IP is,

$$X_{22} - X_{21} = hr \left( \frac{1}{R + \delta} - \frac{1}{R} \right) = \frac{-hr\delta}{R(R + \delta)} = O \left( \frac{1}{R^2} \right),$$

which shows that the perspective effect is of order $R^{-2}$.

SYNTHETIC EXAMPLE

To illustrate the preceding theory we generated in the HP a realization of $N = 300$ point particles within a rectangle from a hard core point process, with the aid of the package Spatstat from the statistical software R (https://www.r-project.org/), see Fig. 3a. Here the HP played the role of a preliminary IP, with origin at the vertical projection $o'(0,0,h = -2)$ of the centre of projection $o(0,0,0)$, see Fig. 1a. With this setup, the containing rectangle was $[-5,15] \times [2,12]$. To estimate the target number $N$, a grid was superimposed on the IP, consisting of square quadrats of side length $t = 1$, with a square fundamental tile of side length $T = 4$. The sampling period was therefore $a'/a = 16$. In Fig. 3a, the observed sample size, namely the total number of particle centres within the quadrats,
is $Q = 20$, whereby Eq. 1 yields $\hat{N} = 320$, with a relative deviation of 6.7%. The empirical coefficient of error $\text{CE}_e(\hat{N}) = \text{SD}_e(\hat{N})/\hat{N} \approx 0.26$ was computed from 1024 automatic Monte Carlo superimpositions of the grid using a systematic resampling design (usually more efficient than independent resampling) – for details see Cruz et al. (2015). This $\text{CE}_e(\hat{N})$ was larger than usual due to the simplicity of the example – in practice the recommended sample size $Q$ to attain a reasonable precision ranges from about 100 to 200 (Cruz and González-Villa, 2018).

The point particles of Fig. 3a were projected
through $\sigma$ onto a vertical circular cylinder of radius $r = 1.4$. After unrolling the cylinder, the transformed IP is represented in Fig. 3b. The superimposition of the (properly scaled) grid of quadrats described above, with the same sampling period of 16, now yielded $CE_{\sigma}(\hat{N}) \approx 0.78$. Thus, the estimation of $\hat{N}$ is in this case about $(78/26)^2 = 9$ times less efficient than in the case illustrated in Fig. 3a. The purpose of this study was to revert the impact of the transform operation induced by the cylindrical projection of the particles by means of a square grid of square quadrats submitted to the same transformation. With the given parameters, and with $\varphi = 0$ and $\varphi = \pi/2$, the relevant Eq. 12 yielded the grid transform illustrated in Fig. 3c. For comparison, two corresponding quadrats from Fig. 3a and Fig. 3c, respectively, are displayed with portions of the forbidden edge (in red) associated with them. It can be readily checked that the individual particle counts are identical for all pairs of corresponding quadrats. In other words, the data collected in Fig. 3a and Fig. 3c, with $Q = 20$ in particular, are identical, and therefore the initial $CE_{\sigma}(\hat{N}) \approx 0.26$ will also hold for Fig. 3c without any further qualification.

**DISCUSSION**

Cruz and González-Villa (2019) implemented the design illustrated in Fig. 3c to estimate the size of a large demonstration from high resolution gigapixel images. Proper computer assisted zooming facilitated individual counting in distant quadrats. In that paper, however, the shape of the transformed quadrats was computed numerically from point landmarks, which affected the efficiency of the procedure. This fact prompted us to treat the model mathematically, showing that the transformed quadrats are bounded by arcs of sinusoidal curves, whose automatic plotting is straightforward.

The projection cylinder used in our model does not have a physical counterpart – it is just a convenient model. Its radius $r$, however, is involved in Eq. 12, which determines the transformed grid. On the other hand, the coordinates of the projection centre, and $h$ in particular, do have a real counterpart because they determine the position of the camera, which is usually not directly available either. In the aforementioned paper, the estimation of the auxiliary parameter $r$, and of the camera coordinates, were estimated indirectly from a system of equations involving the coordinates (obtained with the aid of Google Earth) of physical landmarks available in the scenario of the demonstration. This suggests that the implementation of the design described here on a given gigapixel image may not be that straightforward at present in the absence of the additional information required.

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**REFERENCES**


