PLANAR SECTIONS THROUGH THREE-DIMENSIONAL LINE-SEGMENT PROCESSES

Sascha Djamal Matthes[✉,1] and Dietrich Stoyan[2]

[1]Institut für Keramik, Glas- und Baustofftechnik, Technische Universität Bergakademie Freiberg, D-09596 Freiberg, Germany; [2]Institut für Stochastik, Technische Universität Bergakademie Freiberg, D-09596 Freiberg, Germany
e-mail: Sascha-Djamal.Matthes@ikgb.tu-freiberg.de

ABSTRACT

This paper studies three–dimensional segment processes in the framework of stochastic geometry. The main objective is to find relations between the characteristics of segment processes such as orientation- and length-distribution, and characteristics of their sections with planes. Formulae are derived for the distribution of segment lengths on both sides of the section plane and corresponding orientations, where it is permitted that there are correlations between the angles and lengths of the line-segments.

Keywords: fibre process, fibre-reinforced materials, line-segment process, stereology, stochastic geometry.

INTRODUCTION

Segment processes are stochastic models for random systems of line-segments randomly scattered in space. They belong to the more general class of fibre processes, the mathematical theory of which was developed by Joseph Mecke and coworkers (Mecke and Nagel, 1980; Mecke and Stoyan, 1980b; Chiu *et al.*, 2013).

These processes find important applications in the context of fibre-reinforced materials, where fibres, which can be often modelled as line-segments of negligible thickness, are embedded in a matrix of more or less homogeneous material.

In the now classical papers mentioned above planar sections played an important role. Such sections produce systems of fibre–plane intersection points, which can be statistically analysed with the aim to get information on the spatial fibre system. This setting belongs to the field of stereology, and a classical formula there is

$$L_V = 2N_A \,, \tag{1}$$

where L_V is the mean total fibre length per unit volume and N_A the number of intersection points per unit area. (The formula holds true under the assumptions of statistical homogeneity or stationarity and isotropy, see also Chiu *et al.*, 2013.)

Planar sections through segment processes appear in the context of fibre-reinforced materials, when axial tension is studied. Following Li *et al.* (1991) the intersections of fibres with a plane orthogonal to the tension axis are investigated. Additionally to the characteristics studied when stereology is of interest,

also the residual lengths of the line-segments on both sides of the section plane are of importance in the mechanical calculations. (They have never been considered in the stereological context, since these lengths cannot be measured in the section plane.)

The present paper first explains a natural segment process model, following the pattern of Mecke and Stoyan (1980a). Then it derives formulae for the section process characteristics. Some of them have counterparts in the classical theory, while those related to the residual lengths are new, generalising results in Li *et al.* (1991), who considered the case of segments of constant length. Furthermore, formulae for the maximum and minimum residual segment length are derived since these characteristics play a role in calculations of the contribution of fibres to the mechanical strength in a composite material.

MODEL DESCRIPTION

This paper considers three–dimensional line-segment processes. A realisation of such a process is a set of randomly distributed line-segments in space. To characterise such a line-segment we use its top point (in the sense of the z-axis) $(x,y,z) \in \mathbb{R}^3$, the length $l > 0$ and the angles λ and β denoting azimuth and polar angle of the line-segment. Since we are not interested in the sense of direction of the line-segment, with $\lambda \in [0, 2\pi]$ and $\beta \in \left[0, \frac{\pi}{2}\right]$ a line-segment is well defined.
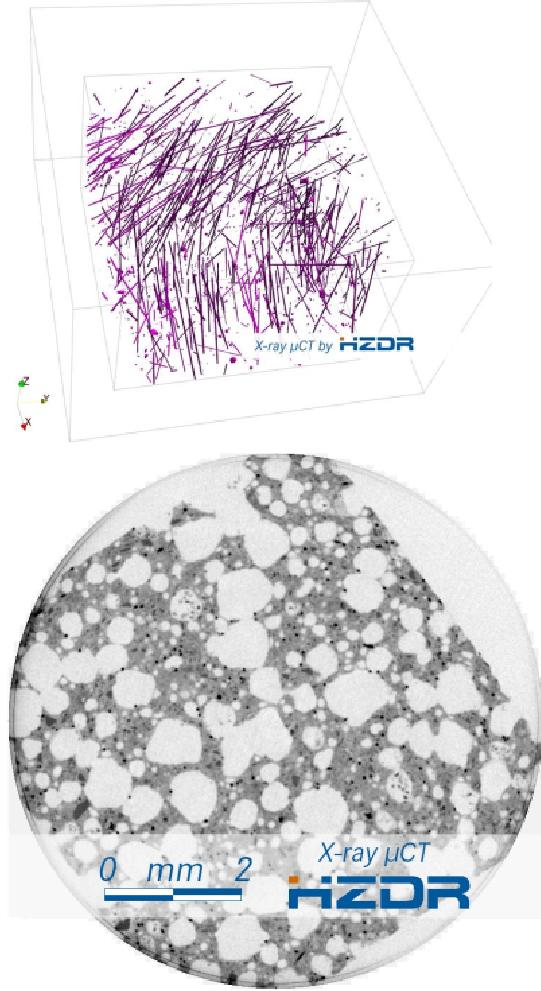
Figure 1. *Computer-tomography-based measurements on fibre-reinforced autoclaved aerated concrete. The upper figure shows the three-dimensional distribution of fibres. The lower figure presents a planar section of reinforced material. The section points of the fibres (black points) form a random point pattern.*
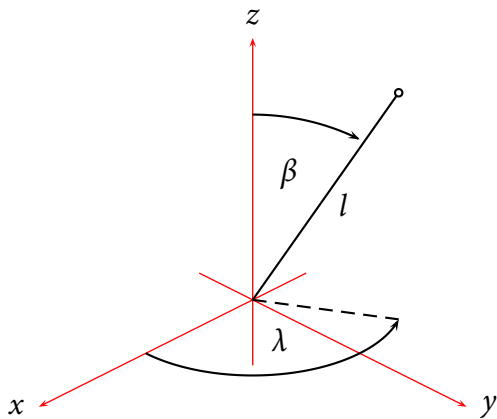

Figure 2. *Geometrical representation of a line-segment which is shifted so that its endpoint meets the origin of the coordinate system.*

A line-segment process is here represented as a

marked point process Ψ_V (for more on marked point processes see Chiu *et al.*, 2013). Realisations of Ψ_V can be written as sequences of marked points:

$$\psi_V = \{[(x_i, y_i, z_i), l_i^V, \lambda_i^V, \beta_i^V]\}, \qquad (2)$$

with $(x_i, y_i, z_i) \in \mathbb{R}^3, l_i^V > 0, \lambda_i^V \in [0, 2\pi]$ and $\beta_i^V \in \left[0, \frac{\pi}{2}\right]$. Moreover we introduce the stochastic variables of the polar angle, B^V, the azimuthal angle, Λ^V, and the fibre length, L^V, of a typical line-segment of Ψ_V.

Here and in the following we assume Ψ_V to be stationary, *i.e.*, the distribution of Ψ_V is translation invariant. It is *not* assumed that the marked point process Ψ_V has some specific distribution, *e.g.*, a marked Poisson process. The results presented in this paper hold for every distribution of a stationary Ψ_V, where B^V and Λ^V are stochastically dependent (see page 57) or independent (see page 60) of L^V.

ESSENTIAL PROPERTIES

The distribution of the marked point process Ψ_V is described by the following characteristics:

Table 1. *Characteristics of the spatial marked point process* Ψ_V

N_V	mean number of top points of line-segments per unit volume
$F_{V,L,B,\Lambda}(l, \beta, \lambda)$	joint distribution function of fibre length L^V and angles B^V and Λ^V of a typical line segment
$F_{V,B}(\beta), F_{V,\Lambda}(\lambda), F_{V,L}(l)$	marginal distribution functions of the stochastic variables B^V, Λ^V and L^V.

The stochastic variables of the polar angle, B^V, and the azimuthal angle, Λ^V may depend on the stochastic variable of the spatial fibre length, L^V.

In order to study the mechanical behavior of fibre-reinforced materials under axial tensions the intersection of a plane with a line-segment system is of peculiar interest. The mechanical effect of a fibre in a homogeneous material depends on the intersection angle and the length of the fibre under and over the plane respectively. Thus these quantities have to be studied. This approach appears in the classical papers by Li *et al.* (1991), Brandt (1985) and in subsequent work. However, in these papers the segment lengths are assumed to be constant.

Due to our homogeneity assumption we choose the intersecting plane to be the (x, y)-plane $S = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Intersection of the segments of Ψ_V with

S yields the marked point process Ψ_A of intersection points with the realizations

$$\psi_A = \{[(\xi_i, \eta_i), r_{1,i}^A, r_{2,i}^A, \lambda_i^A, \beta_i^A]\}, \qquad (3)$$

where (ξ_i, η_i) are the intersection points, $r_{1,i}^A$ and $r_{2,i}^A$ the lengths of the upper and lower part of the segments respectively and λ_i^A and β_i^A the corresponding section angles. Due to stationarity of Ψ_V also the marked point process Ψ_A is stationary.

We introduce the stochastic variables of the polar section angle, B^A, the azimuthal section angle Λ^A and the upper and lower part of the segment which belongs to the typical intersection point, R_1^A and R_2^A respectively. The basic constants and distribution functions of the marked point process Ψ_A are shown in the following table:

Table 2. *Characteristics of the planar marked point process Ψ_A*

N_A	mean number of section points of Ψ_A per unit area
$F_{A,R_1,R_2,\Lambda,B}(r_1, r_2, \lambda, \beta)$	joint distribution function of upper and lower segment lengths and intersection angles B^A, Λ^A at a typical section point
$F_{A,R_1,R_2}(r_1, r_2),$ $F_{A,\Lambda}(\lambda), F_{A,B}(\beta)$	marginal distribution functions of the upper and lower segment lengths and the intersection angles B^A and Λ^A.

BASIC CHARACTERISTICS OF Ψ_A AND RELATIONS TO Ψ_V

MAIN RESULTS

The main objective at this point is to establish relations between the basic constants and distribution functions of Ψ_V and those of Ψ_A. Due to the choice of the intersection plane S and the definition of the azimuthal angle λ, the latter can be ignored.

We concentrate on relations between N_A, N_V and the marginal distribution functions $F_{A,R_1,R_2,B}(r_1, r_2, \beta)$ and $F_{V,L,B}(l, \beta)$. The following general basic equation holds for $N_A, N_V > 0$, $r_1, r_2 > 0$ and $\beta \in \left[0, \frac{\pi}{2}\right]$:

$$N_A F_{A,R_1,R_2,B}(r_1, r_2, \beta) =$$

$$= N_V \int_0^\beta \sin \beta' \int_0^{\min\{r_1, r_2\}} \left(F_{V,L,B}(l + \max\{r_1, r_2\}, \beta')\right.$$
$$\left. - F_{V,L,B}(l, \beta')\right) \mathrm{d}l \, \mathrm{d}\beta'$$
$$+ N_V \cos \beta \int_0^{\min\{r_1, r_2\}} \left(F_{V,L,B}(l + \max\{r_1, r_2\}, \beta)\right.$$
$$\left. - F_{V,L,B}(l, \beta)\right) \mathrm{d}l. \quad (4)$$

The relation of the corresponding probability density functions $f_{V,L,B}(l, \beta)$ and $f_{A,R_1,R_2,B}(r_1, r_2, \beta)$ is therefore

$$N_A f_{A,R_1,R_2,B}(r_1, r_2, \beta) = N_V \cos \beta f_{V,L,B}(r_1 + r_2, \beta). \quad (5)$$

Eq. 4 is the starting point for some important formulae. Let $\beta = \frac{\pi}{2}$ and $r_1, r_2 \to \infty$. Then we obtain the intensity of Ψ_A as

$$N_A = N_V \mathbf{E}(L^V \cos B^V), \qquad (6)$$

where

$$\mathbf{E}(L^V \cos B^V) = \int_0^{\frac{\pi}{2}} \int_0^\infty l \cos \beta f_{V,L,B}(l, \beta) \, \mathrm{d}l \, \mathrm{d}\beta.$$

This expression simplifies for the isotropic case and if L^V and B^V are stochastic independent, see Eq. 26.

With the latter relation we are able to determine $F_{A,R_1,R_2,B}$ if N_V and $F_{V,L,B}$ are given. Furthermore, we obtain with Eq. 4, Eq. 6 and $\beta = \frac{\pi}{2}$ the joint distribution function

$$F_{A,R_1,R_2}(r_1, r_2) = F_{A,R_1,R_2,B}\left(r_1, r_2, \frac{\pi}{2}\right)$$

of the upper and lower segment lengths. These distributions exist if $\mathbf{E}\cos B^V \neq 0$, *i.e.*, if the case of all fibres parallel to the section plane is excluded.

$$N_A F_{A,R_1,R_2}(r_1, r_2) =$$

$$= N_V \int_0^{\min\{r_1, r_2\}} \int_0^{\frac{\pi}{2}} \sin \beta' \left(F_{V,L,B}(l + \max\{r_1, r_2\}, \beta')\right.$$
$$\left. - F_{V,L,B}(l, \beta')\right) \mathrm{d}\beta' \mathrm{d}l,$$

$$F_{A,R_1,R_2}(r_1, r_2) =$$

$$= \frac{1}{\mathbf{E}(L^V \cos B^V)} \int_0^{\min\{r_1, r_2\}} \int_0^{\frac{\pi}{2}} \sin \beta' \cdot$$
$$\left(F_{V,L,B}(l + \max\{r_1, r_2\}, \beta') - F_{V,L,B}(l, \beta')\right) \mathrm{d}\beta' \mathrm{d}l, \quad (7)$$

with $0 \leq r_1, r_2 < \infty$. We can determine the marginal distribution functions for the upper and lower segment lengths

$$F_{A,R_1}(r_1) = \lim_{r_2 \to \infty} F_{A,R_1,R_2}(r_1, r_2)$$

and

$$F_{A,R_2}(r_2) = \lim_{r_1 \to \infty} F_{A,R_1,R_2}(r_1, r_2).$$

The stochastic variables R_1^A and R_2^A are identically distributed with the distribution function

$$F_{A,R}(r) = F_{A,R_1}(r) = F_{A,R_2}(r) =$$
$$\frac{1}{\mathbf{E}(L^V \cos B^V)} \int\limits_0^r \int\limits_0^{\frac{\pi}{2}} \sin\beta' \left(F_{V,B}(\beta')\right.$$
$$\left. - F_{V,L,B}(l,\beta')\right) d\beta' dl \,, \quad (8)$$

for $r > 0$ and $F_{V,B}(\beta) = \lim\limits_{l \to \infty} F_{V,L,B}(l,\beta)$.

With Eqs. 6 and 4 and $r_1, r_2 \to \infty$ we analogously get the distribution function of the section angle B^A

$$F_{A,B}(\beta) =$$
$$\frac{1}{\mathbf{E}(L^V \cos B^V)} \int\limits_0^\infty \left(\int\limits_0^\beta \sin\beta' \left(F_{V,B}(\beta') - F_{V,L,B}(l,\beta')\right) d\beta' \right.$$
$$\left. + \cos\beta \left(F_{V,B}(\beta) - F_{V,L,B}(l,\beta)\right) \right) dl \,.$$
$$(9)$$

With Eqs. 6–9 we have explicit relations for the basic characteristics of Ψ_A. We can add also the probability density functions using Eqs. 6–9:

$$f_{A,R_1,R_2}(r_1, r_2) =$$
$$\frac{1}{\mathbf{E}(L^V \cos B^V)} \int\limits_0^{\frac{\pi}{2}} \cos\beta \, f_{V,L,B}(r_1 + r_2, \beta) d\beta \,, \quad (10)$$

and

$$f_{A,B}(\beta) =$$
$$\frac{1}{\mathbf{E}(L^V \cos B^V)} \int\limits_0^\infty \int\limits_0^\infty \cos\beta \, f_{V,L,B}(r_1 + r_2, \beta) dr_1 dr_2 \,.$$
$$(11)$$

The length of the line-segments were assumed to be stochastically dependent on the angle B^V throughout the above investigations. Eq. 4 shows that, consequently, the line-segment lengths R_1^A and R_2^A are stochastically dependent of the intersection angle B^A.

SKETCH OF THE PROOF OF EQ. 4

The proof follows the pattern of Mecke and Stoyan (1980a), which is also used in Mecke and Stoyan (1980c).

Let $S_D(t_1, t_2, b, c)$ be the expected number of segments $p_i = [(x_i, y_i, z_i), l_i^V, \lambda_i^V, \beta_i^V]$ of the marked point process Ψ_V fulfilling the following conditions:

1. the line-segment p_i intersects a given compact subset D of the plane S,

2. the line-segment has intersection angles $\beta_i^A \in [0, b]$ with $b \in \left[0, \frac{\pi}{2}\right]$ and $\lambda_i^A \in [0, c]$ with $c \in [0, 2\pi]$,

3. the lengths of the segment above and below S fulfil $r_{1,i}^A \leq t_1$ and $r_{2,i}^A \leq t_2$ with $t_1, t_2 > 0$.

The quantity $S_D(t_1, t_2, b, c)$ can be calculated in terms of Ψ_A and Ψ_V, and equating the corresponding terms yields Eq. 4.

The Campbell theorem (see, *e.g.*, Chiu *et al.*, 2013) applied to Ψ_A yields simply

$$S_D(t_1, t_2, b, c) = N_A F_{A,R_1,R_2,B,\Lambda}(t_1, t_2, b, c) \quad (12)$$

for $t_1, t_2 > 0$, $b \in \left[0, \frac{\pi}{2}\right]$ and $c \in [0, 2\pi]$.

In terms of Ψ_V, $S_D(t_1, t_2, b, c)$ can be expressed as

$$S_D(t_1, t_2, b, c) = \mathbf{E}\left(\sum_{p_i \in \Psi_V} f(p_i, t_1, t_2, b, c) \right) \quad (13)$$

for $t_1, t_2 > 0$, $b \in \left[0, \frac{\pi}{2}\right]$ and $c \in [0, 2\pi]$, $p_i = [(x_i, y_i, z_i), l_i^V, \lambda_i^V, \beta_i^V]$ denotes a single segment and f is the indicator function with $f(p_i, t_1, t_2, b, c) = 1$ for segments that fulfil the conditions 1–3 of the definition of S_D, otherwise $f = 0$. In the following we describe the function f.

With $r_{1,i}^A + r_{2,i}^A = l_i^V$ it follows $0 \leq l_i^V \leq t_1 + t_2$ as a first constraint. Furthermore we have $\beta_i^V \in [0, b]$ and $\lambda_i^V \in [0, c]$, therefore we choose

$$f(p_i, t_1, t_2, b, c) =$$
$$\mathbb{1}_{[0,t_1+t_2]}(l_i^V) \mathbb{1}_{[0,b]}(\beta_i^V) \mathbb{1}_{[0,c]}(\lambda_i^V) \mathbb{1}_{Z_0}(x_i, y_i, z_i)$$

for some set $Z_0 = Z_0(\lambda_i^V, \beta_i^V, l_i^V, t_1, t_2)$ with $(x_i, y_i, z_i) \in Z_0$ if and only if p_i fulfils the conditions 1 and 3. Let

$$Z_1 = D \oplus \left\{ s \cdot e_{\lambda_i^V, \beta_i^V}, s \in [0, \min\{l_i^V, t_1\}] \right\}$$

and

$$Z_2 = D \oplus \left\{ s \cdot e_{\lambda_i^V, \beta_i^V}, s \in [l_i^V - \min\{l_i^V, t_2\}, l_i^V] \right\}$$

with $e_{\lambda,\beta} = (\cos\lambda\sin\beta, \sin\lambda\sin\beta, \cos\beta)$, where $\oplus$ denotes the Minkowski addition. Then p_i intersects the set D with an upper segment length $r_{1,i}^A \le t_1$ if and only if $(x_i, y_i, z_i) \in Z_1$ and p_i intersects D with a lower segment length $r_{2,i}^A \le t_2$ if and only if $(x_i, y_i.z_i) \in Z_2$. In both cases p_i intersects D with the intersection angles $\beta_i^V \in [0, b]$ and $\lambda_i^V \in [0, c]$. It follows that p_i fulfils the conditions 1 to 3 if and only if $(x_i, y_i, z_i) \in Z_0 = Z_1 \cap Z_2$. Due to the structure of Z_1 and Z_2 we can write Z_0 as the Minkowski sum of D and a line-segment:

$$Z_0 =$$
$$D \oplus \left\{ s \cdot e_{\lambda_i^V, \beta_i^V}, s \in [l_i^V - \min\{t_2, l_i^V\}, \min\{t_1, l_i^V\}] \right\} .$$
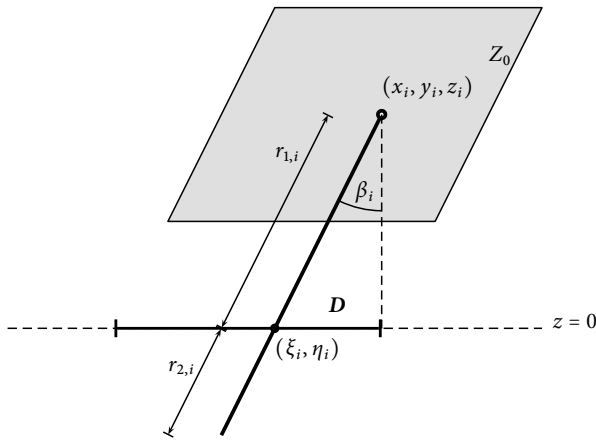
Fig. 3 explains the underlying geometry.

Figure 3. *Underlying geometry in the (x,z)-plane of the intersection of a segment of azimuthal angle $\lambda_i^V = 0$ with the plane S.*

The right hand side of Eq. 13 can be calculated by means of the Campbell theorem. We obtain

$$S_D(t_1, t_2, b, c) =$$
$$= N_V \int\limits_{\mathbb{R}^3} \int\limits_{[0, t_1+t_2] \times [0,b] \times [0,c]} \mathbb{1}_{Z_0}(x, y, z) \cdot$$
$$\mathrm{d}F_{V,L,B,\Lambda}(l, \beta, \lambda)\, \mathrm{d}(x, y, z)$$
$$= N_V \int\limits_{\mathbb{R}^3} \int\limits_0^{t_1+t_2} \int\limits_0^b \int\limits_0^c \mathbb{1}_{Z_0}(x, y, z)$$
$$f_{V,L,B,\Lambda}(l, \beta, \lambda)\, \mathrm{d}\lambda\, \mathrm{d}\beta\, \mathrm{d}l\, \mathrm{d}(x, y, z) .$$
$$(14)$$

By applying Fubini's theorem we can rearrange the order of integration, and we evaluate $\int\limits_{\mathbb{R}^3} \mathbb{1}_{Z_0}(x, y, z)\, \mathrm{d}(x, y, z)$ using Cavalieri's principle. Due

to the representation of Z_0 as a Minkowski sum we can express this integral by the product of the area of D, $v_2(D)$, and the height of Z_0. We get

$$\int\limits_{\mathbb{R}^3} \mathbb{1}_{Z_0}(x, y, z)\mathrm{d}(x, y, z) =$$
$$v_2(D)\cos\beta\left(\min\{l, t_1\} + \min\{l, t_2\} - l\right) .$$

At this point the integrand is independent of λ. With Eq. 12 it follows the equivalence of the marginal distributions

$$F_{A,\Lambda}(\lambda) \equiv F_{V,\Lambda}(\lambda) . \qquad (15)$$

Hence follows, we concentrate on the relation of the marginal distribution functions $F_{A,R_1,R_2,B}$ and $F_{V,L,B}$ and $S_D(t_1, t_2, b, 2\pi)$. We obtain

$$S_D(t_1, t_2, b, 2\pi) =$$
$$N_V \int\limits_{[0, t_1+t_2] \times [0,b]} \cos\beta \cdot$$
$$\left(\min\{l, t_1\} + \min\{l, t_2\} - l\right) \mathrm{d}F_{V,L,B}(l, \beta) .$$
$$(16)$$

Equating Eq. 16 with Eq. 12 and applying integration by parts gives the desired relation Eq. 4.

APPLICATIONS AND DISCUSSION

The stereological formulas Eqs. 6–9 can now be used to verify earlier results and to examine application-related cases.

SUPERPOSITION OF LINE-SEGMENT PROCESSES

In the following example the line-segments follow a special relation between length and direction: long fibres ($l \in [l_0, l_{\max}]$, $0 < l_0 < l_{\max}$) all have the same polar angle $\beta_0 \in [0, \frac{\pi}{2}]$ while short fibres ($l \in [0, l_0]$) are isotropic. The proportion of segments of length $l \in [0, l_0]$ is $p \in [0, 1]$ and for segments with $l \in [l_0, l_{\max}]$ is $1 - p$. The joint probability density function of L^V and B^V is therefore

$$f_{V,L,B}(l, \beta) = \begin{cases} \frac{p}{l_0}\sin\beta , & l \le l_0 \\ \frac{1-p}{l_{\max}-l_0}\delta_{\beta_0}(\beta) , & l_0 \le l \le l_{\max} \\ 0 , & l > l_{\max}. \end{cases}$$
$$(17)$$

Thus, this example represents a superposition of two different line-segment processes. With Eq. 6 the mean number of section points per unit area is

$$N_A = N_V\left(\frac{1}{4}pl_0 + \frac{1}{2}\cos\beta_0(1-p)(l_{\max}+l_0)\right). \quad (18)$$

Moreover using Eq. 7 and Eq. 8 the conditional distribution function

$$F_{A,R_1|R_2 \leq r_2}(r_1) =$$

$$\left(\int_0^{\min\{r_1,r_2\}} \int_0^{\frac{\pi}{2}} \sin\beta' \left(F_{V,L,B}(l + \max\{r_1,r_2\}, \beta') \right. \right.$$

$$\left. \left. - F_{V,L,B}(l,\beta') \right) \mathrm{d}\beta' \mathrm{d}l \right) \Big/$$

$$\left(\int_0^{r_2} \int_0^{\frac{\pi}{2}} \sin\beta' \left(F_{V,B}(\beta') - F_{V,L,B}(l,\beta') \right) \mathrm{d}\beta' \mathrm{d}l \right)$$

as well as the conditional density function

$$f_{A,R_1|R_2 \leq r_2}(r_1) =$$

$$\frac{\int_0^{\frac{\pi}{2}} \sin\beta' \left(F_{V,L,B}(r_1 + r_2, \beta') - F_{V,L,B}(r_1, \beta) \right) \mathrm{d}\beta'}{\int_0^{r_2} \int_0^{\frac{\pi}{2}} \sin\beta' \left(F_{V,B}(\beta') - F_{V,L,B}(l,\beta') \right) \mathrm{d}\beta' \mathrm{d}l}$$

can be calculated. The conditional probability density function $f_{A,R_1|R_2 \leq r_2}(r_1)$ is shown in Fig. 4 for $l_0 = 1$, $l_{max} = 3$, $\beta_0 = \frac{3}{4}$ and $p = \frac{3}{4}$ and by means of Eq. 18 the ratio $\frac{N_A}{N_V}$ is 0.553. Thus, N_A can be determined once N_V is known.
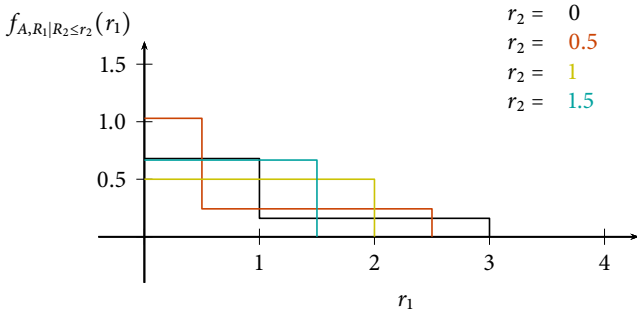

Figure 4. *The conditional probability density function $f_{A,R_1|R_2 \leq r_2}(r_1)$ of the upper segment length R_1^A under the condition the lower segment length R_2^A is bounded from above.*

INDEPENDENT FIBRE ANGLES AND LENGTHS

Assume the line-segment length L^V is stochastically independent of the angles Λ^V and B^V for the line-segment process Ψ_V. Under this assumption this line-segment process can be characterized using the following parameters and functions:

Table 3. *Characteristics of the spatial marked point process Ψ_V under the assumption that L^V and B^V are stochastically independent.*

N_V	mean number of top points of line segments per unit volume
$F_{V,L}(l)$	distribution function of length of a typical segment of Ψ_V
$F_{V,\Lambda}(\lambda), F_{V,B}(\beta)$	distribution function of azimuth and polar angle of a typical segment of Ψ_V

Eq. 4 simplifies in the following way:

$$N_A F_{A,R_1,R_2,B}(r_1,r_2,\beta) =$$

$$= N_V \int_0^{\min\{r_1,r_2\}} \int_0^{\beta} \left(F_{V,L}(l + \max\{r_1,r_2\}) - F_{V,L}(l) \right)$$

$$\cdot \cos\beta' \, \mathrm{d}F_{V,B}(\beta') \, \mathrm{d}l$$

$$= N_V \int_0^{\min\{r_1,r_2\}} \left(F_{V,L}(l + \max\{r_1,r_2\}) - F_{V,L}(l) \right) \mathrm{d}l$$

$$\cdot \int_0^{\beta} \sin\beta' F_{V,B}(\beta') \, \mathrm{d}\beta'$$

for $r_1, r_2 > 0$, $\beta \in \left[0, \frac{\pi}{2}\right]$ and $N_A, N_V > 0$. It follows that the segment lengths R_1^A, R_2^A and the polar angle B^A of the segment process Ψ_A are stochastically independent.

We therefore concentrate on the marginal distribution functions $F_{A,R_1,R_2}(r_1,r_2)$ and $F_{A,B}(\beta)$. We have

$$N_A F_{A,R_1,R_2}(r_1,r_2) F_{A,B}(\beta) =$$

$$N_V \int_0^{\min\{r_1,r_2\}} \left(F_{V,L}(l + \max\{r_1,r_2\}) - F_{V,L}(l) \right) \mathrm{d}l \cdot$$

$$\int_0^{\beta} \sin\beta' F_{V,B}(\beta') \, \mathrm{d}\beta'. \quad (19)$$

Using this simplified relation we obtain

$$N_A = N_V \mathbf{E} L^V \mathbf{E} \cos B^V, \quad (20)$$

$$F_{A,R_1,R_2}(r_1,r_2) =$$

$$\frac{1}{\mathbf{E} L^V} \int_0^{\min\{r_1,r_2\}} \left(F_{V,L}(l + \max\{r_1,r_2\}) - F_{V,L}(l) \right) \mathrm{d}l$$

$$\quad (21)$$

and

$$F_{A,B}(\beta) = \frac{1}{\mathbf{E}\cos B^V} \int\limits_0^\beta \sin\beta' F_{V,B}(\beta') \, d\beta' . \qquad (22)$$

The segment lengths R_1^A and R_2^A are identically distributed, *i.e.*, the marginal distributions F_{A,R_1} and F_{A,R_2} are equivalent:

$$F_{A,R}(r) = F_{A,R_1}(r) = F_{A,R_2}(r) =$$
$$\frac{1}{\mathbf{E}L^V} \int\limits_0^r \left(1 - F_{V,L}(l)\right) dl , \quad r > 0 . \quad (23)$$

Concerning the corresponding probability density functions of Ψ_V and Ψ_A it holds

$$f_{A,R_1,R_2}(r_1, r_2) = \frac{1}{\mathbf{E}L^V} f_{V,L}(r_1 + r_2) , \qquad (24)$$

and

$$f_{A,B}(\beta) = \frac{1}{\mathbf{E}\cos B^V} f_{V,B}(\beta)\cos\beta . \qquad (25)$$

THE ISOTROPIC CASE

We assume that the line-segment length L^V is stochastically independent of the angles Λ^V and B^V for the line-segment process Ψ_V. Furthermore the line-segment process Ψ_V is assumed to be isotropic, *i.e.*, the directional vector of the typical line-segment is uniformly distributed on the unit sphere. We have

$$f_{V,B}(\beta) = \sin\beta ,$$

and the distribution function

$$F_{V,B}(\beta) = 1 - \cos\beta$$

for this case. Eq. 9 yields

$$f_{A,B}(\beta) = \sin 2\beta ,$$

and

$$F_{A,B}(\beta) = \sin^2\beta ,$$

which is a result true for general fibre processes, see Mecke and Nagel (1980).

With Eq. 6 we obtain N_A, the mean number of section points per unit area as

$$N_A = \frac{1}{2} N_V \mathbf{E}L^V . \qquad (26)$$

This result is a special case of another well-known stereological formula, namely (11.3.3) in Chiu *et al.* (2013), where characteristics of germ-grain models are studied. Here the "grain" is an isotropic line-segment.

SEGMENTS OF CONSTANT LENGTH

In papers such as Li *et al.* (1991) and Brandt (1985) line-segment processes and their sections with planes are studied. They derived formulas for the strength of fibre-reinforced materials using calculations for segments of constant length. If in this context the constant length is chosen to be $l_0 > 0$, we obtain with Eq. 8 the marginal distribution function $F_{A,R}(r)$. It holds $F_{V,L}(l) = \Theta(l - l_0)$ with the Heaviside function

$$\Theta(x) = \left\{ \begin{array}{ll} 0, & x < 0 \\ 1, & x \geq 0 \end{array} \right.$$

and

$$F_{A,R}(r) = \frac{1}{l_0} \int\limits_0^r \left(1 - \Theta(l - l_0)\right) dl$$
$$= \left\{ \begin{array}{ll} \frac{r}{l_0}, & r \leq l_0 \\ 1, & r \geq l_0 \end{array} \right.$$

for the distribution function of the upper and lower segment length at a typical section point. This distribution is the uniform distribution on $[0, l_0]$. We therefore obtain the mean values of the residual segment lengths $\mathbf{E}R_1^A = \mathbf{E}R_2^A = \frac{1}{2} l_0$. This result coincides with results of Li *et al.* (1991).

Moreover the line-segments in (Li *et al.*, 1991) are considered to be isotropic. Using Eq. 26 we have $N_A = \frac{1}{2} N_V l_0$.

A PARAMETRIC MODEL

In Chin *et al.* (1988), Hegler (1985) and other papers line-segment length and angle distribution functions for segment processes Ψ_V with L^V independent of B^V appear such as

$$F_{V,L}(l) = 1 - e^{-(ml)^k}, \quad m, k, l > 0 ,$$

the Weibull distribution function, and

$$F_{V,B}(\beta) = \frac{1 - \exp(-\eta\beta)}{1 - \exp\left(-\frac{\pi}{2}\eta\right)}, \quad \beta \in \left[0, \frac{\pi}{2}\right], \eta > 0 .$$

These distribution functions are motivated by the process of manufacture of some fibre-reinforced material. They model different effects such as breakage of fibres before moulding or a preferred orientation of fibres after moulding, where the shape parameter η models the orientation density of the line-segments. A large η indicates a major preferential alignment of the line-segments in the z-direction, see also Kacir *et al.* (1975) and Chin *et al.* (1988). Note that this parametric model does not include the

isotropic case. The Figs. 5 and 6 give an impression of the influence of the parameters m, k and η on the probability density functions $f_{V,L}$ and $f_{V,B}$.
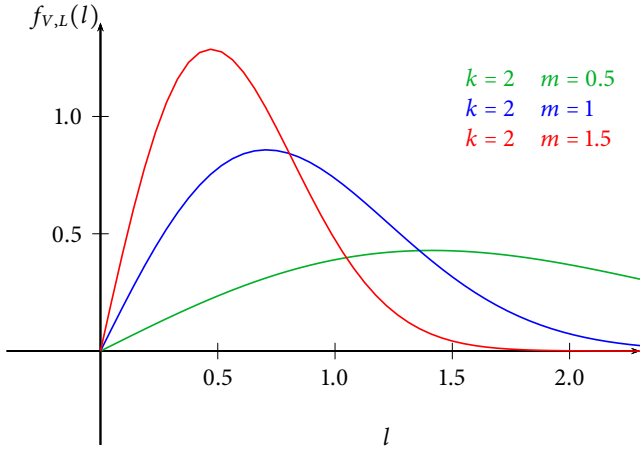

Figure 5. *Influence of the parameters k and m on the Weibull probability densitiy function of L^V.*
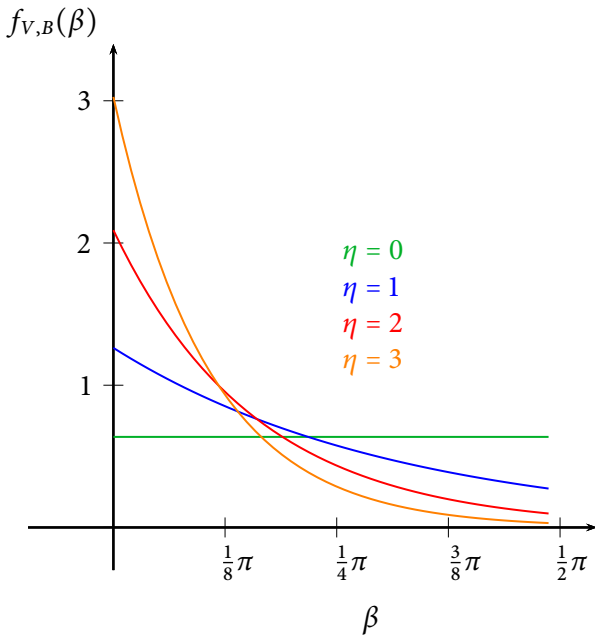

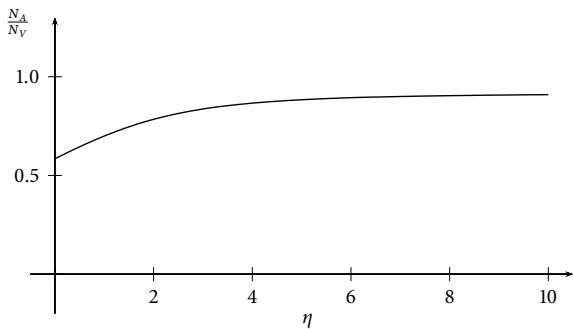Figure 6. *Influence of the shape parameter η on the probability density function of polar angle B^V.*

Figure 7. *The ratio $\frac{N_A}{N_V}$ in dependence on shape parameter η.*
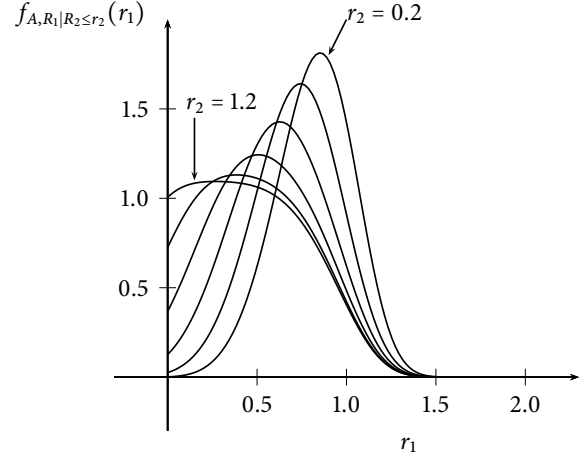

Figure 8. *The conditional probability density function $f_{A,R_1|R_2 \leq r_2}(r_1)$ in dependence on r_2 for Weibull-distributed fibre lengths. This is the probability density function of the length R_1^A of the segment above the plane S, if the segment length R_2^A below S is bounded from above. Here r_2 ranges from 0.2 to 1.2 with a step size of 0.2.*

The distribution of the residual segment lengths can be easily computed using the Eqs. 6–9. For the parameters $m = 1$ and $k = 5$ the results are shown in Figs. 7–9.

Fig. 7 show that $\frac{N_A}{N_V}$ goes to a limit value as η tends to infinity. This limit is $\mathbf{E}L^V$, the mean length of the line-segments (in the case $m = 1$ and $k = 5$ we have the limit value $\mathbf{E}L^V \approx 0.918$). This is plausible since for large η the segments have a preferred orientation in z-direction. Therefore with $\eta \to \infty$ all line-segments have the polar angle $B^V = 0$ and therefore $B^A = 0$. Applying this case to Eq. 6 we find $N_A = N_V \mathbf{E}L^V$.

Furthermore, Fig. 8 shows that the conditional distribution of the upper segment length

$$f_{A,R_1|R_2 \leq r_2}(r_1) = \frac{F_{V,L}(r_1 + r_2) - F_{V,L}(r_1)}{\int_0^{r_2} \left(1 - F_{V,L}(l)\right) \mathrm{d}l} \quad (27)$$

is more concentrated if the lower segment length is fixed at small values. Moreover it can be shown that $f_{A,R_1|R_2 \leq r_2}(r_1)$ coincides with $f_{V,L}(l)$ if $r_2 \to 0$. Thus with decreasing upper bound of R_2^A the conditional probability density function in Fig. 8 tends to the probability density function of the Weibull distribution with $m = 1$ and $k = 5$ in Fig. 8. With increasing r_2 the conditional probability $f_{A,R_1|R_2 \leq r_2}(r_1)$ tends to $\frac{1}{\mathbf{E}L^V} \left(1 - F_{V,L}(r_1)\right)$. Since the segment lengths R_1^A and R_2^A are independent of the intersection angle B^A in this case the conditional probability density function $f_{A,R_1|R_2 \leq r_2}$ is independent of η.
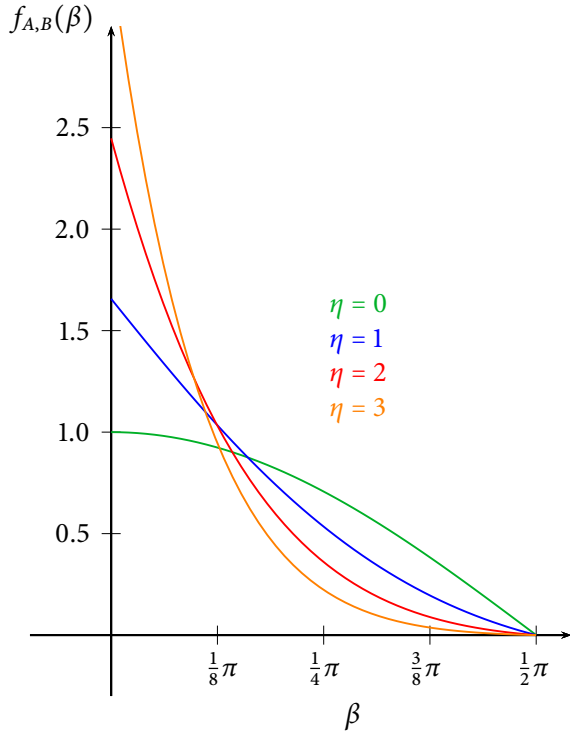
Figure 9. *The probability density function $f_{A,B}(\beta)$ of intersection angles in dependence on orientation distribution parameter η which varies from 0 to 3.*

In Fig. 9 we see that with increasing η the segments at the typical section point show a preferred direction in the z-axis. Since the segment lines tend to have a small polar angle B^V for large η, the segments intersecting S have a small polar angle B^A too. For $\eta \to 0$ the probability density function $f_{A,B}(\beta)$ tends to $\cos\beta$. Note that this is not the isotropic case since for isotropic line-segments it holds $f_{A,B}(\beta) = \sin 2\beta$.

MAXIMUM AND MINIMUM RESIDUAL SEGMENT LENGTH

We investigate the stochastic variables $M = \max\{R_1^A, R_2^A\}$ and $m = \min\{R_1^A, R_2^A\}$, which are only in special cases stochastic independent. We therefore consider first the joint distribution function $F_{A,m,M}(r_m, r_M)$ with $r_m, r_M > 0$. Furthermore we consider the marginal distribution functions

$$F_{A,M}(r) = P(M \le r) = P(R_1^A \le r, R_2^A \le r) \quad \text{and}$$

$$F_{A,m}(r) = P(m \le r) = 1 - P(R_1^A > r, R_2^A > r)$$

for $r > 0$. Using simple ideas of probability we obtain

$$F_{A,m,M}(r_m, r_M) = \begin{cases} \begin{aligned} &F_{A,R_1,R_2}(r_m, r_M) \\ &+ F_{A,R_1,R_2}(r_M, r_m) \\ &- F_{A,R_1,R_2}(r_m, r_m) \end{aligned} & r_m \le r_M \\ \\ F_{A,R_1,R_2}(r_M, r_M) & r_M \le r_m \end{cases}$$

$$F_{A,M}(r) = F_{A,R_1,R_2}(r, r)$$

and

$$F_{A,m}(r) = 2F_{A,R}(r) - F_{A,R_1,R_2}(r, r) .$$

With Eqs. 7 and 8 we can relate these distribution functions with the distribution functions of Ψ_V:

$$F_{A,m,M}(r_m, r_M) =$$

$$\frac{1}{\mathbf{E}L^V} \int_0^{r_m} \left(2F_{V,L}(l + r_M) - F_{V,L}(l + r_m) - F_{V,L}(l) \right) \mathrm{d}l , \tag{28}$$

$$F_{A,M}(r) =$$

$$\frac{1}{\mathbf{E}L^V} \int_0^r \left(F_{V,L}(l + r) - F_{V,L}(l) \right) \mathrm{d}l , \tag{29}$$

$$F_{A,m}(r) =$$

$$\frac{1}{\mathbf{E}L^V} \int_0^r \left(2 - F_{V,L}(l) - F_{V,L}(l + r) \right) \mathrm{d}l . \tag{30}$$

The marginal distributions simplify with $F_{V,\overline{L}}(l) = 1 - F_{V,L}(l)$:

$$F_{A,M}(r) = \frac{1}{\mathbf{E}L^V} \int_0^r \left(F_{V,\overline{L}}(l) - F_{V,\overline{L}}(l + r) \right) \mathrm{d}l$$

and

$$F_{A,m}(r) = \frac{1}{\mathbf{E}L^V} \int_0^r \left(F_{V,\overline{L}}(l) + F_{V,\overline{L}}(l + r) \right) \mathrm{d}l .$$

In case of Weibull-distributed lengths it holds $F_{V,L}(l) = 1 - \mathrm{e}^{-(ml)^k}$ and $F_{V,\overline{L}}(l) = \mathrm{e}^{-(ml)^k}$. The corresponding results are sketched in Fig. 10.
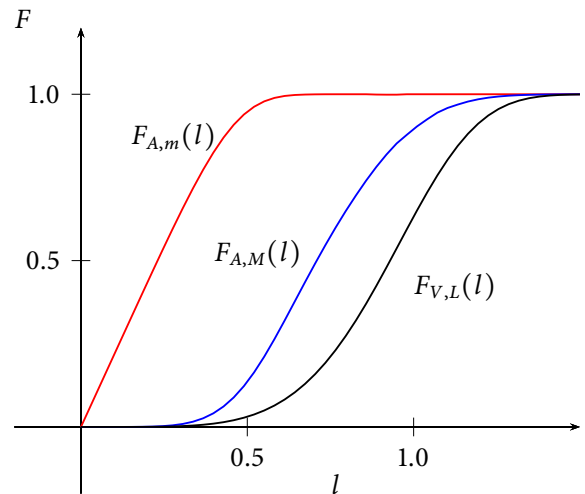

Figure 10. *Distribution functions of minimum and maximum residual length, $F_{A,m}(l)$ and $F_{A,M}(l)$, for Weibull-distributed total lengths with $m = 1$ and $k = 5$.*

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