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Review article

ADVANCES IN MULTIDIMENSIONAL SIZE THEORY

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ABSTRACT

Size Theory was proposed in the early 90's as a geometrical/topological approach to the problem of Shape Comparison, a very lively research topic in the fields of Computer Vision and Pattern Recognition. The basic idea is to discriminate shapes by comparing shape properties that are provided by continuous functions valued in $\mathbb{R}$, called measuring functions and defined on topological spaces associated to the objects to be studied. In this way, shapes can be compared by using a descriptor named size function, whose role is to capture the features described by measuring functions and represent them in a quantitative way. However, a common scenario in applications is to deal with multidimensional information. This observation has led to considering vector-valued measuring functions, and consequently the multidimensional extension of size functions, namely the k-dimensional size functions. In this work we survey some recent results about size functions in this multidimensional setting, with particular reference to the localization of their discontinuities.

Keywords: multidimensional size function, shape analysis, size theory, topological persistence.

INTRODUCTION

Shape Analysis and Comparison are probably two of the most challenging issues in the fields of Computer Vision, Computer Graphics, Image Analysis and Pattern Recognition. Shape models are characterized by a considerable amount of visual, semantic and digital data, and therefore the development of methods able to extract the most relevant properties of a shape is necessary when dealing with such an information. Recently, an increasing interest has been devoted to methods deriving from Topological Persistence, giving relevance to consider the topological features of a shape with respect to some geometrical properties conveyed by real functions defined on the shape itself (Frosini and Landi, 1999; Carlsson *et al.*, 2005; Cohen-Steiner *et al.*, 2005). In this context, Size Theory was introduced in the early 90's as a geometrical/topological approach to the problem of Shape Analysis and Comparison, studying the concept of *size function*, a mathematical tool able to capture the qualitative aspects of a shape and represent them in a quantitative way. More precisely, the main idea in Size Theory is to model a shape by means of a topological space $\mathscr{M}$, endowed with a continuous function φ called *measuring function*. Such a function is chosen according to applications and describes the features considered relevant for shape characterization. In this way, the *size pair* $(\mathscr{M},\varphi)$ can be seen as a representation of a given shape with respect to the properties expressed by the selected measuring function φ. Part of the qualitative information contained in $(\mathscr{M},\varphi)$ is then quantitatively stored in the associated size function $\ell_{(\mathscr{M},\varphi)}$, describing some topological attributes that persist in the sublevel sets of $\mathscr{M}$ induced by φ. Following this approach, comparing two shapes can be reduced to the simpler comparison of the associated size functions, making use of a suitable distance as, *e.g.*, the *matching distance* (d'Amico *et al.*, 2003; 2006; 2010). In the context of Algebraic Topology, an analogous notion to the one of size function has been developed under the name of *size homotopy group* (Frosini and Mulazzani, 1999).

More recently, similar ideas have been re-proposed by Persistent Homology according to a homological approach (Edelsbrunner *et al.*, 2002; Edelsbrunner and Harer, 2008). In this setting, the concept of size function coincides with the dimension of the 0-th multidimensional persistent homology group, *i.e.*, the 0-th rank invariant (Carlsson and Zomorodian, 2007).

Since their introduction, size functions have been extensively studied and applied to concrete problems in the fields of Computer Vision and Graphics, Image Analysis and Pattern Recognition, with particular reference to the 1-dimensional setting, *i.e.*, to the case of measuring functions taking values in $\mathbb{R}$ (Verri *et al.*, 1993; Uras and Verri, 1997; Dibos *et al.*, 2004; Cerri *et al.*, 2006; Biasotti *et al.*, 2008c). Similarly, Persistence Homology was initially developed in a 1-dimensional version (*i.e.*, studying the topological evolution of a one-parameter increasing family of spaces), with applications in shape description (Carlsson *et al.*, 2005), hole detection in sensor network (de Silva and Ghrist, 2007) and data simplification (Bubenik and Kim, 2007).

However, a common scenario in applications is to deal with multidimensional information: This can be easily understood if we consider, *e.g.*, the representation of color in the RGB model. Other similar examples can be found in the context of computational biology, in medical environments, as well as in scientific simulations of natural phenomena. These observations had led to pay close attention to the study of Topological Persistence in a multidimensional setting (Frosini and Mulazzani, 1999; Biasotti *et al.*, 2007; Carlsson and Zomorodian, 2007; Biasotti *et al.*, 2008a; Edelsbrunner and Harer, 2008; Ghrist, 2008; Carlsson, 2009). Referring to Size Theory, the term *multidimensional* is related to considering vector-valued measuring functions, and consequently the multidimensional extension of size functions, namely the *k-dimensional size functions*.

In this paper we review some recent results concerning the theory of size functions associated to measuring functions taking values in $\mathbb{R}^k$ (Biasotti *et al.*, 2007; 2008a), with particular reference to the study of their structure and to the localization of their discontinuities (Cerri and Frosini, 2008). Indeed, this last research line is a necessary step toward the development of efficient algorithms for the computation of multidimensional size functions and their application to concrete problems.

MULTIDIMENSIONAL SIZE THEORY

In this section we introduce the basic definitions and results about size functions, confining ourselves to those we consider relevant to the survey purposes of this paper. For further details about Size Theory, the reader is referred to Frosini and Mulazzani (1999); Biasotti *et al.* (2007; 2008a;b).

The main idea underlying the notion of (k-dimensional) size function is to study a given shape by performing a topological exploration of a suitable topological space $\mathscr{M}$, with respect to some geometric properties provided by an $\mathbb{R}^k$-valued continuous function $\vec{\varphi} = (\varphi_1, \ldots, \varphi_k)$ defined on $\mathscr{M}$. Under these assumptions, the *size function* $\ell_{(\mathscr{M},\vec{\varphi})}$ is then a stable and compact descriptor of the topological changes occurring in the lower level sets $\{P \in \mathscr{M} : \varphi_i(P) \leq t_i, i = 1, \ldots, k\}$ as $\vec{t} = (t_1, \ldots, t_k)$ varies in $\mathbb{R}^k$.

In the classical formulation of Size Theory, $\mathscr{M}$ is required to be a non-empty, compact and locally connected Hausdorff space, and $\vec{\varphi} : \mathscr{M} \to \mathbb{R}^k$ is a continuous function. However, since some of the results we are going to present imply differential considerations, for the sake of simplicity we prefer *here* to restrict our hypothesis, by assuming that $\mathscr{M}$ is a closed C^1 Riemannian manifold, endowed with a C^1 function $\vec{\varphi} = (\varphi_1, \ldots, \varphi_k) : \mathscr{M} \to \mathbb{R}^k$.

In the context of Size Theory, any pair $(\mathscr{M}, \vec{\varphi})$, with $\mathscr{M}$ and $\vec{\varphi} = (\varphi_1, \ldots, \varphi_k) : \mathscr{M} \to \mathbb{R}^k$ satisfying the previous assumptions, is called a *size pair*. The function $\vec{\varphi}$ is said to be a *k-dimensional measuring function*. We define the following relations $\preceq$ and $\prec$ in $\mathbb{R}^k$: for $\vec{x} = (x_1, \ldots, x_k)$ and $\vec{y} = (y_1, \ldots, y_k)$, we shall write $\vec{x} \preceq \vec{y}$ (resp. $\vec{x} \prec \vec{y}$) if and only if $x_i \leq y_i$ (resp. $x_i < y_i$) for every index $i = 1, \ldots, k$. Moreover, $\mathbb{R}^k$ will be endowed with the usual max-norm: $\|(x_1, x_2, \ldots, x_k)\|_\infty = \max_{1 \leq i \leq k} |x_i|$. Now we are ready to introduce the concept of size function for a size pair $(\mathscr{M}, \vec{\varphi})$. The open set $\{(\vec{x}, \vec{y}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{x} \prec \vec{y}\}$ will be denoted by Δ^+. For every k-tuple $\vec{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, we shall define the set $\mathscr{M}\langle \vec{\varphi} \preceq \vec{x} \rangle$ as $\{P \in \mathscr{M} : \varphi_i(P) \leq x_i,\ i = 1, \ldots, k\}$.

Definition 1.1. We call the *(k-dimensional) size function* associated with the size pair $(\mathscr{M}, \vec{\varphi})$ the function $\ell_{(\mathscr{M},\vec{\varphi})} : \Delta^+ \to \mathbb{N}$, defined by setting $\ell_{(\mathscr{M},\vec{\varphi})}(\vec{x}, \vec{y})$ equal to the number of connected components in the set $\mathscr{M}\langle \vec{\varphi} \preceq \vec{y} \rangle$ containing at least one point of $\mathscr{M}\langle \vec{\varphi} \preceq \vec{x} \rangle$.

Remark 1.2. The concept of size function is strongly related to the ones of persistent homology group and rank invariant (Edelsbrunner *et al.*, 2002; Carlsson and Zomorodian, 2007). More precisely, the (multidimensional) size function $\ell_{(\mathscr{M},\vec{\varphi})}$ coincides with the 0-th rank invariant associated with the (multi)filtration induced on $\mathscr{M}$ by $\vec{\varphi}$. For a formal definition of rank invariant the reader is referred to Carlsson and Zomorodian (2007).

In what follows, the case of measuring functions taking value in $\mathbb{R}^k$ will be addressed by using the term "k-dimensional".

Example 1.3 (*The particular case* $k = 1$**).** Close attention should be paid to the particular framework of measuring functions taking values in $\mathbb{R}$, *i.e.*, to the 1-dimensional case. Indeed, Size Theory has been widely developed in this setting (Biasotti *et al.*, 2008b), proving that each 1-dimensional size function admits a compact representation as a formal series of points and lines of $\mathbb{R}^2$ (Frosini and Landi, 2001). As a consequence of this peculiar structure, a suitable *matching distance* between 1-dimensional size functions can be easily introduced, showing the stability of these descriptors with respect to such a distance (d'Amico *et al.*, 2003; 2010). All these properties make the concept of 1-dimensional size function central in the approach to the k-dimensional case proposed in Biasotti *et al.* (2008a).

According to the notations used in the literature about the case $k = 1$, the symbols $\vec{\varphi}$, $\vec{x}$, $\vec{y}$ will be replaced respectively by φ, x, y.

When referring to a (1-dimensional) measuring function $\varphi : \mathcal{M} \to \mathbb{R}$, the size function $\ell_{(\mathcal{M},\varphi)}$ associated with $(\mathcal{M},\varphi)$ contains information about the pairs $(\mathcal{M}\langle\varphi \le x\rangle, \mathcal{M}\langle\varphi \le y\rangle)$, where $\mathcal{M}\langle\varphi \le t\rangle$ is defined by setting $\mathcal{M}\langle\varphi \le t\rangle = \{P \in \mathcal{M} : \varphi(P) \le t\}$ for $t \in \mathbb{R}$.

Before going on, we observe that for $k = 1$, the domain Δ^+ of a size function reduces to be the open subset of the real plane given by $\{(x,y) \in \mathbb{R}^2 : x < y\}$.

Fig. 1 shows an example of a size pair $(\mathcal{M},\varphi)$ together with the size function $\ell_{(\mathcal{M},\varphi)}$. On the left (Fig. 1$(a)$) the considered size pair $(\mathcal{M},\varphi)$ can be found, where $\mathcal{M}$ is the curve drawn by a solid line, and φ is the ordinate function. On the right (Fig. 1(b)) the associated 1-dimensional size function $\ell_{(\mathcal{M},\varphi)}$ is depicted.

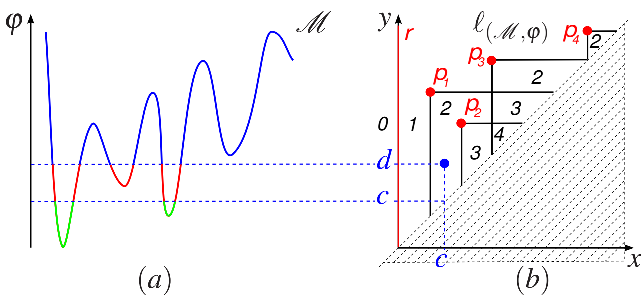

Fig. 1. (a) *The topological space* $\mathcal{M}$ *and the measuring function* φ. (b) *The associated size function* $\ell_{(\mathcal{M},\varphi)}$.

As can be seen, the domain $\Delta^+ = \{(x,y) \in \mathbb{R}^2 : x < y\}$ is divided into regions by solid lines. These lines represent the discontinuities of $\ell_{(\mathcal{M},\varphi)}$, which are located by the following theorem:

Theorem 1.4. *Let* $\mathcal{M}$ *be a closed* C^1 *Riemannian manifold, and let* $\varphi : \mathcal{M} \to \mathbb{R}$ *be a* C^1 *measuring function. If* $(\bar{x},\bar{y})$ *is a discontinuity point for* $\ell_{(\mathcal{M},\varphi)}$*, then either* $\bar{x}$ *or* $\bar{y}$ *or both are critical values for* φ*.*

Each region of Δ^+ is labeled by a number, coinciding with the constant value that $\ell_{(\mathcal{M},\varphi)}$ takes in the interior of that region. For example, let us compute the value of $\ell_{(\mathcal{M},\varphi)}$ at the point (c,d). By Definition 1.1 in the case $k = 1$, it is sufficient to count how many of the three connected components in the sublevel $\mathcal{M}\langle\varphi \le d\rangle$ contain at least one point of $\mathcal{M}\langle\varphi \le c\rangle$. It can be easily checked that $\ell_{(\mathcal{M},\varphi)}(c,d) = 2$.

Due to their typical structure, it has been proved that the information conveyed by a 1-dimensional size function can be combinatorially stored in a formal series of points and lines (Frosini and Landi, 2001). Roughly speaking, this can be done by observing that each 1-dimensional size function is representable by means of a linear combination (with natural numbers as coefficients) of characteristic functions associated to triangles, possibly unbounded, lying on the domain Δ^+. Indeed, the bounded triangles are of the form $\{(x,y) \in \Delta^+ : \alpha \le x < y < \beta\}$, while the unbounded ones are of the form $\{(x,y) \in \Delta^+ : \eta \le x < y)\}$. Hence, a simple and compact representation can be provided if one considers the formal series obtained by associating a triangular set given by $\{(x,y) \in \Delta^+ : \alpha \le x < y < \beta\}$ to the point (α,β), and a triangular set given by $\{(x,y) \in \Delta^+ : \eta \le x < y)\}$ to the point at infinity $(\eta,+\infty)$. The points of a formal series having finite coordinates are called *proper cornerpoints*, while the ones with a coordinate at infinity are named *cornerpoints at infinity* or *cornerlines*. For example, the size function $\ell_{(\mathcal{M},\varphi)}$ shown in Fig. 1(b) admits the representation by formal series given by $r + p_1 + p_2 + p_3 + p_4$, where r is the only cornerpoint at infinity, with coordinates $(0,+\infty)$.

According to the 1-dimensional setting, the problem of comparing two size pairs can be easily translated into the simpler one of comparing sets of points, via the representation by formal series of the associated 1-dimensional size functions. In d'Amico *et al.* (2003; 2010), the *matching distance* d_{match} has proven to be a suitable distance between these descriptors. In plain words, the matching distance d_{match} measures the cost of moving the points of a formal series onto the points of another one, with respect to the max-norm. An application of d_{match} is shown in Fig. 2(c).

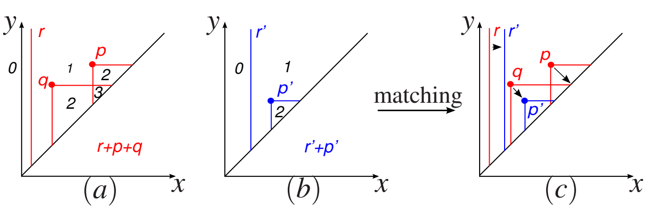

Fig. 2. (a) *The size function corresponding to the formal series* $r + p + q$. (b) *The size function corresponding to the formal series* $r' + p'$. (c) *The matching between the two formal series, realizing the matching distance between the two size functions.*

As can be seen in Fig. 2, different 1-dimensional size functions may in general have a different number of cornerpoints. Therefore d_{match} allows a proper cornerpoint to be matched to a point of the diagonal: this matching can be interpreted as the deletion of a proper cornerpoint. Moreover, we stress that the d_{match} has proven to be stable with respect to perturbations of the measuring functions (d'Amico *et al.*, 2003; 2010), making this framework suitable when coping with applications in Shape Comparison. For a formal definition and further details about the matching distance the reader is referred to d'Amico *et al.* (2006; 2010).

Remark 1.5. The notion of d_{match} is strictly related to the ones of bottleneck distance, used in Cohen-Steiner *et al.* (2005) to prove the stability of persistence diagrams, and Hausdorff distance. More precisely, d_{match} reduces to be the bottleneck distance under the restriction that the measuring functions are *tame* (we recall that a continuous real function $f : \mathscr{M} \to \mathbb{R}$ is tame if it has a finite number of homological critical values and the homology groups $H_k(f^{-1}(-\infty,a])$ are finite-dimensional for all $k \in \mathbb{Z}$ and $a \in \mathbb{R}$). The matching distance reduces to be the Hausdorff distance when considering left- and right-total relations instead of bijections between cornerpoints.

REDUCTION TO THE 1-DIMENSIONAL CASE

In this section we review the approach to the k-dimensional extension of size functions proposed in Biasotti *et al.* (2008a). In that work, the authors show that the case $k > 1$ can be reduced to the 1-dimensional setting by a change of variable and the use of a suitable foliation. In particular, they prove that a parameterized family of half-planes in $\mathbb{R}^k \times \mathbb{R}^k$ can be given, such that the restriction of a k-dimensional size function $\ell_{(\mathscr{M},\vec{\varphi})}$ to each of these half-planes turns out to be a particular 1-dimensional size function. This approach finds motivations in the fact that generalizing to an arbitrary dimension (*i.e.*, to the case $k > 1$) the concepts of proper cornerpoint and cornerpoint at infinity seems not to be trivial. We recall that these notions, defined in the case of 1-dimensional size functions, play a central role in the introduction of the representation by formal series. Consequently, at a first glance it seems not possible to provide the multidimensional analogue of the matching distance d_{match} and therefore it is not clear how to obtain stability under perturbations of the measuring functions. On the other hand, all these problems can be overcome via the results we are going to survey.

Before proceeding, we need to introduce some further notation.

For every unit vector $\vec{l} = (l_1,\ldots,l_k)$ of $\mathbb{R}^k$ such that $l_i > 0$ for $i = 1,\ldots,k$, and for every vector $\vec{b} = (b_1,\ldots,b_k)$ of $\mathbb{R}^k$ such that $\sum_{i=1}^k b_i = 0$, the pair $(\vec{l},\vec{b})$ is said to be *admissible*. The set of all admissible pairs in $\mathbb{R}^k \times \mathbb{R}^k$ is denoted by Adm_k. Given an admissible pair $(\vec{l},\vec{b})$, the half-plane $\pi_{(\vec{l},\vec{b})}$ of $\mathbb{R}^k \times \mathbb{R}^k$ is defined by the following parametric equations:

$$\pi_{(\vec{l},\vec{b})} : \begin{cases} \vec{x} = s\vec{l} + \vec{b} \\ \vec{y} = t\vec{l} + \vec{b} \end{cases},$$

for $s,t \in \mathbb{R}$, with $s < t$.

Remark 2.1. It can be easily proved that the collection of half-planes $\left\{\pi_{(\vec{l},\vec{b})} : (\vec{l},\vec{b}) \in Adm_k\right\}$ is a foliation of Δ^+, hence for every point of the domain Δ^+ there exists one and only one half-plane $\pi_{(\vec{l},\vec{b})}$, with $(\vec{l},\vec{b}) \in Adm_k$, containing the point itself. Moreover, the half-plane $\pi_{(\vec{l},\vec{b})}$ depends continuously on the pair $(\vec{l},\vec{b})$.

We are now ready to present the main result in the approach to the multidimensional case proposed in Biasotti *et al.* (2008a):

Theorem 2.2 (Reduction Theorem)**.** *Let $(\vec{l},\vec{b})$ be an admissible pair, and let $F^{\vec{\varphi}}_{(\vec{l},\vec{b})} : \mathscr{M} \to \mathbb{R}$ be defined by setting*

$$F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) = \max_{i=1,\ldots,k} \left\{ \frac{\varphi_i(P) - b_i}{l_i} \right\}.$$

Then, for every $(\vec{x},\vec{y}) = (s\vec{l} + \vec{b}, t\vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$ the following equality holds:

$$\ell_{(\mathscr{M},\vec{\varphi})}(\vec{x},\vec{y}) = \ell_{(\mathscr{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}(s,t).$$

In the following, we shall use the symbol $F^{\vec{\varphi}}_{(\vec{l},\vec{b})}$ in the sense of the Reduction Theorem 2.2.

Roughly speaking, the Reduction Theorem 2.2 states that, on each half-plane of the foliation, the restriction of a given multidimensional size function coincides with a particular size function in two scalar variables, *i.e.*, a 1-dimensional one. A first important consequence is the possibility of representing a multidimensional size function $\ell_{(\mathscr{M},\vec{\varphi})}$ by a collection of formal series of points and lines, following the machinery described in Example 1.3 for the case $k = 1$. Indeed, each admissible pair $(\vec{l},\vec{b})$ can be associated with a formal series $\sigma_{(\vec{l},\vec{b})}$ describing the 1-dimensional size function $\ell_{(\mathscr{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$. Therefore, on each half-plane $\pi_{(\vec{l},\vec{b})}$ the matching distance between 1-dimensional size functions can be applied, showing that it is stable with respect to perturbations of the multidimensional measuring functions and to the choice of the leaves of the foliation (Biasotti *et al.*, 2008a, Prop. 2 and 3). These stability properties lead to the definition of a distance $D_{\text{match}}(\ell_{(\mathscr{M},\vec{\varphi})},\ell_{(\mathscr{N},\vec{\psi})})$ between two multidimensional size functions $\ell_{(\mathscr{M},\vec{\varphi})}$ and $\ell_{(\mathscr{N},\vec{\psi})}$, given by $D_{\text{match}}(\ell_{(\mathscr{M},\vec{\varphi})},\ell_{(\mathscr{N},\vec{\psi})}) = \sup_{(\vec{l},\vec{b})\in Adm_k} \min_{i=1,\dots,k} l_i \cdot d_{\text{match}}(\ell_{(\mathscr{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})},\ell_{(\mathscr{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})})$ (Biasotti *et al.*, 2008a, Def. 8).

Remark 2.3. Let us observe that choosing a non-empty and finite subset $A \subseteq Adm_k$, and substituting $\sup_{(\vec{l},\vec{b})\in Adm_k}$ with $\max_{(\vec{l},\vec{b})\in A}$ in the definition of $D_{\text{match}}(\ell_{(\mathscr{M},\vec{\varphi})},\ell_{(\mathscr{N},\vec{\psi})})$, we obtain a computable pseudodistance between k-dimensional size functions, that is stable and hence suitable for applications.

Before going on, we now provide an example showing how the Reduction Theorem 2.2 can be applied for comparing k-dimensional size functions.

Example 2.4. In $\mathbb{R}^3$ take $\mathscr{Q} = [-1,1]\times[-1,1]\times[-1,1]$ and the sphere $\mathscr{S}$ of equation $x^2+y^2+z^2=1$. Let also $\vec{\Phi}:\mathbb{R}^3\to\mathbb{R}^2$ be the C^1 function taking each point (x,y,z) to the pair (x^2,z^2). Now consider the size pairs $(\mathscr{M},\vec{\varphi})$ and $(\mathscr{N},\vec{\psi})$, where $\mathscr{M}$ is the "smoothed version" of $\partial\mathscr{Q}$ represented in Fig. 3, $\mathscr{N}=\mathscr{S}$ and $\vec{\varphi}$, $\vec{\psi}$ are the restrictions of $\vec{\Phi}$ to $\mathscr{M}$ and $\mathscr{N}$, respectively. In order to compare the (2-dimensional) size functions $\ell_{(\mathscr{M},\vec{\varphi})}$ and $\ell_{(\mathscr{N},\vec{\psi})}$, we are interested in studying the foliation in half-planes $\pi_{(\vec{l},\vec{b})}$, where $\vec{l}=(\cos\theta,\sin\theta)$ with $0<\theta<\pi/2$, and $\vec{b}=(a,-a)$ with $a\in\mathbb{R}$. Any such half-plane is given by

$$\begin{cases} x_1 = s\cos\theta + a\\ x_2 = s\sin\theta - a\\ y_1 = t\cos\theta + a\\ y_2 = t\sin\theta - a \end{cases},$$

with $s,t\in\mathbb{R}$, $s<t$. Fig. 3 shows the size functions $\ell_{(\mathscr{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ and $\ell_{(\mathscr{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}$, for $\theta=\pi/4$ and $a=0$, *i.e.*, $\vec{l}=(\sqrt{2}/2,\sqrt{2}/2)$ and $\vec{b}=(0,0)$. In this case we obtain $F^{\vec{\varphi}}_{(\vec{l},\vec{b})}=\sqrt{2}\max\{\varphi_1,\varphi_2\}=\sqrt{2}\max\{x^2,z^2\}$ and $F^{\vec{\psi}}_{(\vec{l},\vec{b})}=\sqrt{2}\max\{\psi_1,\psi_2\}=\sqrt{2}\max\{x^2,z^2\}$. Therefore, the Reduction Theorem 2.2 implies that, for every $(x_1,x_2,y_1,y_2)\in\pi_{(\vec{l},\vec{b})}$, we have

$$\begin{aligned}\ell_{(\mathscr{M},\vec{\varphi})}(x_1,x_2,y_1,y_2) &= \ell_{(\mathscr{M},\vec{\varphi})}\left(\frac{s}{\sqrt{2}},\frac{s}{\sqrt{2}},\frac{t}{\sqrt{2}},\frac{t}{\sqrt{2}}\right)\\ &= \ell_{(\mathscr{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}(s,t),\\ \ell_{(\mathscr{N},\vec{\psi})}(x_1,x_2,y_1,y_2) &= \ell_{(\mathscr{N},\vec{\psi})}\left(\frac{s}{\sqrt{2}},\frac{s}{\sqrt{2}},\frac{t}{\sqrt{2}},\frac{t}{\sqrt{2}}\right)\\ &= \ell_{(\mathscr{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}(s,t)\ .\end{aligned}$$

The matching distance $d_{\text{match}}(\ell_{(\mathscr{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})},\ell_{(\mathscr{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})})$ equals $\sqrt{2}-\varepsilon-(1/\sqrt{2})=\sqrt{2}/2-\varepsilon$ ($1\gg\varepsilon>0$, ε depending on the "smoothness level" of $\mathscr{M}$), *i.e.*, the cost of moving the point of coordinates $(0,\sqrt{2}-\varepsilon)$ onto the point of coordinates $(0,1/\sqrt{2})$, computed with respect to the max-norm. The points $(0,\sqrt{2}-\varepsilon)$ and $(0,1/\sqrt{2})$ are representative of the characteristic triangles of the size functions $\ell_{(\mathscr{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ and $\ell_{(\mathscr{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}$, respectively. Note that the pseudodistance we obtained from D_{match} (*cf.* Remark 2.3) by computing the matching distance d_{match} for $\vec{l}=(\sqrt{2}/2,\sqrt{2}/2)$ and $\vec{b}=(0,0)$, equals to $\frac{\sqrt{2}}{2}\cdot(\frac{\sqrt{2}}{2}-\varepsilon)$. This implies that, even by considering just one half-plane of the foliation, it is possible to effectively compare multidimensional size functions. Let us conclude by observing that $\ell_{(\mathscr{M},\varphi_1)}\equiv\ell_{(\mathscr{N},\psi_1)}$ and $\ell_{(\mathscr{M},\varphi_2)}\equiv\ell_{(\mathscr{N},\psi_2)}$. In other words, the multidimensional size functions, with respect to $\vec{\varphi},\vec{\psi}$, are able to discriminate the cube and the sphere, while both the 1-dimensional size functions, with respect to φ_1,φ_2 and ψ_1,ψ_2, cannot do that. The higher discriminatory power of multidimensional size functions provides a further motivation for their introduction and use.

The comparison procedure based on Theorem 2.2 and illustrated in Example 2.4 is the core of the machinery developed for concrete applications in the context of Shape Analysis. For example, in Biasotti *et al.* (2007) k-dimensional size functions are used for comparing and retrieving 2- and 3-dimensional data, using both vectorial (*i.e.*, triangle meshes) and raster (voxel images) representations. Indeed, in that work the authors consider two different databases of 280 surface models and of 420 volume models, respectively. In order to compare and retrieve the surface models, each of them is equipped with the same 2-dimensional measuring function, computing the pseudodistance induced by D_{match} (*cf.* Remark 2.3) between the related 2-dimensional size functions over 4 different half-planes of the foliation described in Example 2.4. The same approach is used to compare the volume models, but choosing a 3-dimensional measuring function instead of a 2-dimensional one, and computing the restrictions of the outcoming 3-dimensional size functions over a single half-plane of $\Delta^+\subseteq\mathbb{R}^3\times\mathbb{R}^3$. The promising results obtained in both the applications suggest that Multidimensional Size Theory can be effectively used to analyze and compare 3D digital shapes (represented by surface or volume models) equipped by vector-valued functions.

For further details about the experimental results described here see Biasotti *et al.* (2007). Other experiments can be found in Biasotti *et al.* (2008a).

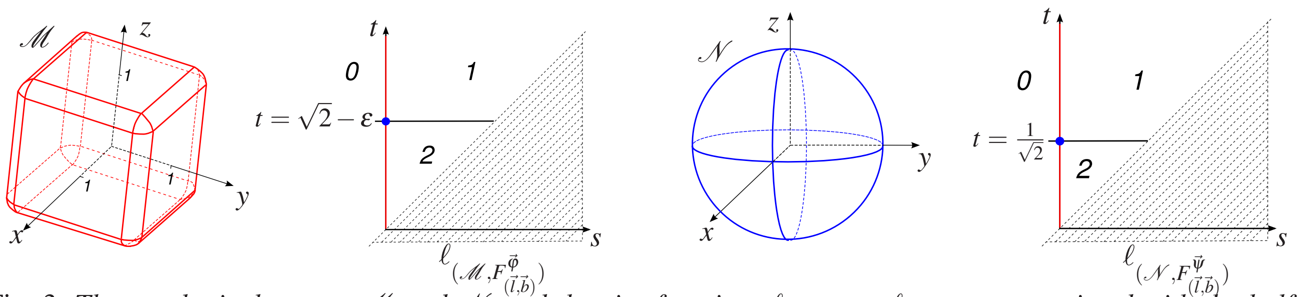

Fig. 3. *The topological spaces* $\mathscr{M}$ *and* $\mathscr{N}$ *and the size functions* $\ell_{(\mathscr{M}, F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathscr{N}, F^{\vec{\psi}}_{(\vec{l},\vec{b})})}$ *associated with the half-plane* $\pi_{(\vec{l},\vec{b})}$*, for* $\vec{l} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ *and* $\vec{b} = (0,0)$.

DISCONTINUITIES IN THE MULTIDIMENSIONAL CASE

The approach to the case $k > 1$ reviewed in the previous section has revealed to be useful both in applying multidimensional size functions to concrete problems and in solving some questions related to their intrinsic structure. Indeed, the theoretical machinery introduced in Biasotti *et al.* (2008a) has been used in a recent work in order to study the localization of the discontinuities for multidimensional size functions. More precisely, in Cerri and Frosini (2008) it has been proved that a generalization of Theorem 1.4 holds when $k > 1$, giving a necessary condition for a point $(\vec{x}, \vec{y}) \in \Delta^+$ to be a discontinuity point for a k-dimensional size function $\ell_{(\mathscr{M}, \vec{\varphi})}$. In this section we review the main considerations leading to this result, which is stated in Theorem 3.3. For further details the reader is referred to Cerri and Frosini (2008).

Consider the size pair $(\mathscr{M}, \vec{\varphi})$ and the associated multidimensional size function $\ell_{(\mathscr{M}, \vec{\varphi})}$. From now to Theorem 3.3 an admissible pair $(\vec{l}, \vec{b}) \in Adm_k$ will be fixed and the 1-dimensional size function $\ell_{(\mathscr{M}, F)}$ will be considered, where $F(P) = \max_{i=1,\dots,k}\{(\varphi_i(P) - b_i)/l_i\}$ for all $P \in \mathscr{M}$. The functions F and $\ell_{(\mathscr{M}, F)}$ will be said the (1-dimensional) measuring function and the size function corresponding to the half-plane $\pi_{(\vec{l},\vec{b})}$, respectively.

In what follows, the symbol $\ell_{(\mathscr{M}, \vec{\varphi})}(\cdot, \vec{y})$ will denote the function taking each k-tuple $\vec{x} \prec \vec{y}$ to the value $\ell_{(\mathscr{M}, \vec{\varphi})}(\vec{x}, \vec{y})$. An analogous convention will hold for the symbol $\ell_{(\mathscr{M}, \vec{\varphi})}(\vec{x}, \cdot)$.

The first step toward claiming Theorem 3.3 consists in the observation that a slightly modified version of Theorem 1.4 holds for the 1-dimensional size function $\ell_{(\mathscr{M}, F)}$ associated to the half-plane $\pi_{(\vec{l},\vec{b})}$. Indeed, such an adaptation is due to the fact that the 1-dimensional measuring function F is, in general, not C^1. The idea is then to introduce an approximation of F by a sequence of C^1 functions (F_p). In this way, Theorem 1.4 can be applied, getting a differential necessary condition, depending on the half-plane $\pi_{(\vec{l},\vec{b})}$, for the discontinuity points of the functions $\ell_{(\mathscr{M}, F_p)}$. Due to the stability properties of the matching distance d_{match} between 1-dimensional size functions, it is possible to prove that the differential condition passes to the limit $p \to +\infty$, and therefore it also holds for the discontinuity points of $\ell_{(\mathscr{M}, F)}$.

This first result can then be extended to the discontinuities of the multidimensional size function $\ell_{(\mathscr{M}, \vec{\varphi})}$. Indeed, in Cerri and Frosini (2008) it is shown that a correspondence exists between the discontinuity points of $\ell_{(\mathscr{M}, F)}$ and the ones of $\ell_{(\mathscr{M}, \vec{\varphi})}$. This can be proved by applying the Reduction Theorem 2.2 and the stability of d_{match} with respect to the choice of the half-planes foliating Δ^+.

Finally, the result given in Theorem 3.3 refines the differential necessary condition obtained for the discontinuity points of $\ell_{(\mathscr{M}, \vec{\varphi})}$, by removing the dependance on the foliation of Δ^+. In order to do this, in Cerri and Frosini (2008) the following definitions of *pseudocritical point* and *pseudocritical value* for a vector-valued C^1 function have been used:

Definition 3.1. Let $\vec{\chi} : \mathscr{M} \to \mathbb{R}^h$ be a C^1 function. A point $P \in \mathscr{M}$ is said to be a *pseudocritical point for* $\vec{\chi}$ if the convex hull of the gradients $\nabla \chi_i(P)$, $i = 1, \dots, h$, contains the null vector, *i.e.*, there exist $\lambda_1, \dots, \lambda_h \in \mathbb{R}$ such that $\sum_{i=i}^{h} \lambda_i \cdot \nabla \chi_i(P) = \mathbf{0}$, with $0 \le \lambda_i \le 1$ and $\sum_{i=1}^{h} \lambda_i = 1$. If P is a pseudocritical point of $\vec{\chi}$, then $\vec{\chi}(P)$ will be called a *pseudocritical value for* $\vec{\chi}$.

Remark 3.2. Definition 3.1 corresponds to the Fritz John necessary condition for optimality in Nonlinear Programming (Bazaraa *et al.*, 1993). For further references see Smale (1973). The concept of the pseudocritical point is strongly related also to the ones of Jacobi Set (Edelsbrunner and Harer, 2002) and generalized gradient (Clarke, 1983). In literature, pseudocritical points are also called Pareto-critical points.

Roughly speaking, Definition 3.1 states that a point $P \in \mathscr{M}$ is pseudocritical for the function $\vec{\chi} : \mathscr{M} \to \mathbb{R}^h$ if, moving from P, it is not possible to "choose a direction" on $\mathscr{M}$ allowing one to decrease *at the same time* each component of $\vec{\chi}(P)$ (with respect to a first order approximation of $\vec{\chi}$). According to Definition 3.1 and considering a suitable projection $\rho : \mathbb{R}^k \to \mathbb{R}^h$, with $\rho(\vec{x}) = (x_{i_1}, \ldots, x_{i_h})$ for some indices $i_1, \ldots, i_h$, the next theorem has been proved in Cerri and Frosini (2008), locating the discontinuity points of $\ell_{(\mathscr{M},\vec{\varphi})}$ and avoiding any reference to the half-planes $\pi_{(\vec{l},\vec{b})}$:

Theorem 3.3. *Let $(\vec{x},\vec{y}) \in \Delta^+$ be a discontinuity point for $\ell_{(\mathscr{M},\vec{\varphi})}$. Then at least one of the following statements holds:*

(i) *$\vec{x}$ is a discontinuity point for $\ell_{(\mathscr{M},\vec{\varphi})}(\cdot,\vec{y})$ and then a projection ρ exists such that $\rho(\vec{x})$ is a pseudocritical value for $\rho \circ \vec{\varphi}$;*

(ii) *$\vec{y}$ is a discontinuity point for $\ell_{(\mathscr{M},\vec{\varphi})}(\vec{x},\cdot)$ and then a projection ρ exists such that $\rho(\vec{y})$ is a pseudocritical value for $\rho \circ \vec{\varphi}$.*

In other words, the result claimed in Theorem 3.3 states that a discontinuity point for a multidimensional size function has at least one pseudocritical coordinate up to a suitable projection, under the hypothesis that the considered measuring function is C^1. We observe that this result implies several relevant consequences. First, it contributes to clarifying the structure and simplifying the computation of multidimensional size functions. In order to explain this point let us consider the case of a compact smooth manifold $\mathscr{M}$ endowed with a smooth function $\vec{\varphi} = (\varphi_1, \varphi_2) : \mathscr{M} \to \mathbb{R}^2$. It is immediate to verify that all pseudocritical points belong to the Jacobi set of $\vec{\varphi}$, that is the set where the gradients $\nabla\varphi_1$ and $\nabla\varphi_2$ are parallel. This implies (Edelsbrunner and Harer, 2002) that in the generic case the pseudocritical points belong to a 1-submanifold $\mathscr{J}$ of $\mathscr{M}$ (in local coordinates such a manifold is determined by the vanishing of the Jacobian of $\vec{\varphi}$). For the computation of $\mathscr{J}$ we refer to Edelsbrunner and Harer (2002). Now, let $\mathscr{P}$ be the set of pseudocritical values for $\vec{\varphi}$, and let $\mathscr{C}_1$ (respectively $\mathscr{C}_2$) be the set of critical values for φ_1 (resp. φ_2). Following these notations, if we assume that $\mathscr{A}_1 = \mathscr{C}_1 \times \mathbb{R}^3$, $\mathscr{A}_2 = \mathbb{R} \times \mathscr{C}_2 \times \mathbb{R}^2$, $\mathscr{B}_1 = \mathbb{R}^2 \times \mathscr{C}_1 \times \mathbb{R}$, $\mathscr{B}_2 = \mathbb{R}^3 \times \mathscr{C}_2$, $\mathscr{P}_1 = \mathscr{P} \times \mathbb{R}^2$ and $\mathscr{P}_2 = \mathbb{R}^2 \times \mathscr{P}$, then Theorem 3.3 allows us to claim that all discontinuity points (x_1, x_2, y_1, y_2) of the size function $\ell_{(\mathscr{M},\vec{\varphi})}$ belong to the set $\mathscr{K} = \Delta^+ \cap (\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{P}_1 \cup \mathscr{P}_2)$.

In the light of this new information, we can imagine the possibility of constructing new algorithms to efficiently compute multidimensional size functions. Let us consider the connected components in which the domain of $\ell_{(\mathscr{M},\vec{\varphi})}$ is divided by the set $\mathscr{K}$. Since size functions are locally constant at each point of continuity (we recall that they are natural-valued), we immediately obtain that $\ell_{(\mathscr{M},\vec{\varphi})}$ is constant at each of those connected components. It follows that the computation of $\ell_{(\mathscr{M},\vec{\varphi})}$ just requires the computation of its value at only one point for each connected component. These observations open the way to new and more efficient methods of computation for multidimensional size functions.

Our results also make new pseudodistances between size pairs computable in an easier way. Let us provide a simple example. Consider the two size pairs $(\mathscr{M},\vec{\varphi})$, $(\mathscr{N},\vec{\psi})$ introduced in Example 2.4. Let also $\mathscr{P}_{\vec{\varphi}}$ (respectively $\mathscr{P}_{\vec{\psi}}$) be the set of pseudocritical values for $\vec{\varphi}$ (resp. $\vec{\psi}$). It can be easily verified that $\mathscr{P}_{\vec{\varphi}}$ and $\mathscr{P}_{\vec{\psi}}$ are respectively the subsets of $\mathbb{R}^2$ represented in Fig. $4(a)$ and Fig. $4(b)$. It is trivial to check that the Hausdorff distance between $\mathscr{P}_{\vec{\varphi}}$ and $\mathscr{P}_{\vec{\psi}}$ approximates the value $\frac{1}{\sqrt{2}}$ (the approximation depending on the "smoothness level" of $\mathscr{M}$), thus giving a measure of the (dis)similarity between $(\mathscr{M},\vec{\varphi})$ and $(\mathscr{N},\vec{\psi})$.

CONCLUSIONS

In this paper we surveyed recent advances in the theory of multidimensional size functions, spanning the main results leading to their application to concrete problems in the fields of Computer Vision and Graphics, Image Analysis and Pattern Recognition. Close attention has been paid also to the review of the most interesting theoretical properties concerning these shape descriptors, with particular reference to the localization of their discontinuities. Indeed, this last research line appears to be promising in improving the computation and the use of multidimensional size functions in the context of concrete applications.

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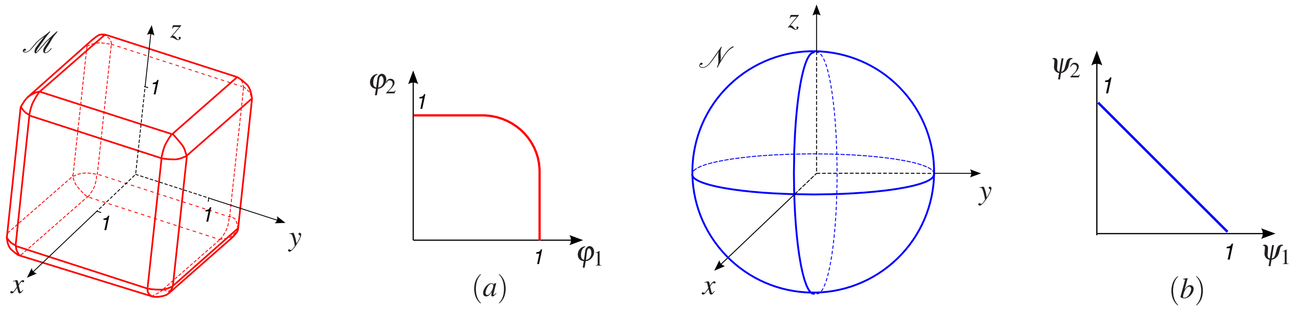

Fig. 4. *The topological spaces $\mathscr{M}$, $\mathscr{N}$ and the pseudocritical values for the functions $\vec{\varphi}$ (a) and $\vec{\psi}$ (b).*

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